"CAN YOU EVEN?" AN EXPLORATION OF POLYOMINOS

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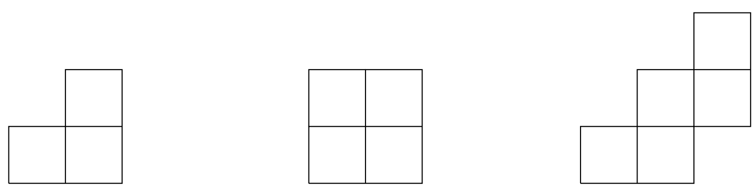
Abstract. This paper was inspired by AMS Mathematics Magazine problem #2175 in [1] which asks "For which integers $n \geq 3$ can an $n \times n$ square grid be colored black and white—using each color at least once—so that every possible placement of a W-pentomino covers an even number of black squares?" We extend this problem to consider other and more general polyominos in square grids, as well as grids composed of triangles, and present their respective results. We also prove a solution to the original problem that is $2 \leq n \leq 5$.

Introduction and Nomenclature

The study of polyominos became popular with the emergence of the game Tetris, where players attempt to interlock Tetris tiles—what we refer to as "polyominos"—and avoid gaps to score points. Figure 1 shows an example of three simple polyominos.



(a) V-triomino **(b)** S-tetromino **(c)** W-pentomino

Figure 1. The V-triomino, S-tetromino, and W-pentomino

This paper focuses on generalizations and extensions of the problem proposed in the abstract. A general polyomino with any number of cells can be defined as follows.

Definition 1. (Polyomino) A **polyomino** X is a contiguous arrangement of cells. See Figure 1 for examples of simple polyominos.

We will also define some other terms relevant to the problem.

Definition 2. (p-grid) A **p-grid** δ is a subset of the two-dimensional plane tiled by congruent p polygons. Each smallest p polygon is referred to as a cell.

A coordinate system will enable us to identify individual cells. For a rectangular grid, we use standard Cartesian coordinates, where $(1, 1)$ represents the bottom left square. In other grids, such as the triangular grid T_n, we may define alternative coordinate systems.

Definition 3. (Coloring) For a grid δ, let a **coloring** C be an assignment of a color $c : \delta \to \{0, 1\}$ to each cell, where 1 is black and 0 is white.

Definition 4. (X-sufficient coloring) A coloring is **X-sufficient** if it utilizes both black and white, and all possible placements and orientations of an X-polyomino will cover an even number of black squares (which includes 0). It is **X-insufficient** otherwise.

Definition 5. (X-sufficient grid) A grid is **X-sufficient** if it has an X-sufficient coloring. It is **X-insufficient** otherwise.

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Conventional polyominos on $n \times n$ square grids

The following results concern two containment arguments for proving X-sufficiency or X-insufficiency on larger square grids given a respective smaller square grid. These lemma will be useful later throughout this paper.

Lemma 1. *If for some polyomino X and sets $A \subset B$ of cells, B is X-sufficient, then A is X-sufficient.*

Proof. Consider the coloring of the cells of A when B is colored such that the conditions of X-sufficiency in Definition 4 hold for B. Then, as $A \subset B$, any placement of an X-polyomino on A will cover an even number of black squares. Thus, A is X-sufficient. ∎

The contrapositive yields the following.

Corollary 2. *If for some polyomino X and sets $A \subset B$ of cells, A is X-insufficient, then B is X-insufficient.*

One particularly useful implication of these lemmas is that if a $k \times k$ grid is insufficient, then all $n \times n$ grids are insufficient for $n \geq k$. Note that Lemma 1 and Corollary 2 apply to general p-grids (as opposed to only square ones) as well.

The V-triomono. As the simplest case of our problem, the V-triomino presents an elementary application of Corollary 2.

Proposition 3. *The coloring of S_n for $n \geq 2$ is V-insufficient.*

Proof. When $n = 2$, there are only 6 possible colorings of S_2 up to rotational symmetry, as seen in Figure 2. By inspection, we can deduce that S_2 is V-insufficient.

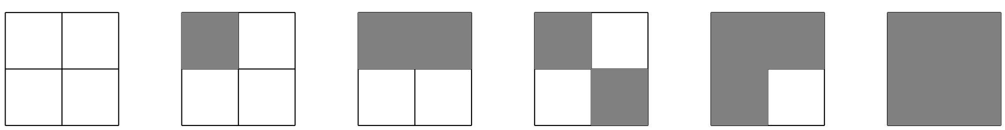


Figure 2. Six possible colorings of S_2 up to rotational symmetry.

Notice that the first and last colorings are V-insufficient because they do not utilize both colors. For $n > 2$, we note that $S_2 \subset S_n$. Thus, by Corollary 2, we have that S_n is V-insufficient for $n \geq 2$. ∎

The 2×2 Square S_2-tetromino. The next consideration of polyominos is naturally square tetrominos, or what we refer to as the S_2-tetromino. We present a proof that the $n \times n$ square grid is S_2-sufficient.

Proposition 4. *For all $n \geq 2$, the $n \times n$ square grid is S_2-sufficient.*

Proof. We will provide two S_2-sufficient colorings for an $n \times n$ grid, where $n > 1$.

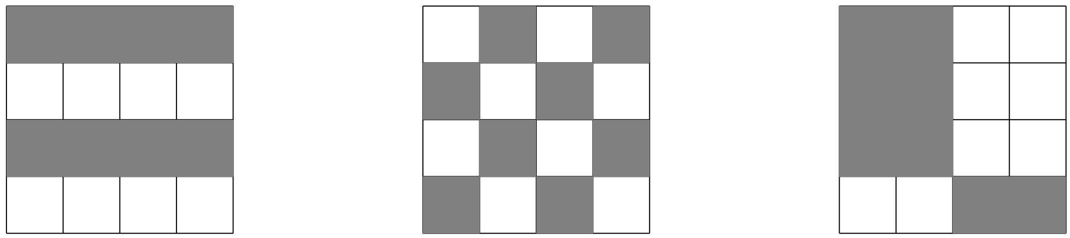
As an S_2-tetromino necessarily contains two vertically adjacent cells and thus two consecutive y-coordinates, it must contain two cells with even y-coordinates and two with odd y-coordinates. Therefore, it suffices to color all cells with even y-coordinates black, so that every possible S_2-tetromino covers exactly 2 black cells.

An alternative method is to color only cells with x- and y-coordinates of the same parity. Each placement of an S_2-tetromino necessarily contains exactly two of these cells, and so the same result is achieved.

Therefore, for all $n \geq 2$, the $n \times n$ grid is S_2-sufficient. ∎

We show an example.

Example. Below are three cases of colorings that are S_2-sufficient.



(a) even y **(b)** (x, y) same parity **(c)** another sufficient coloring

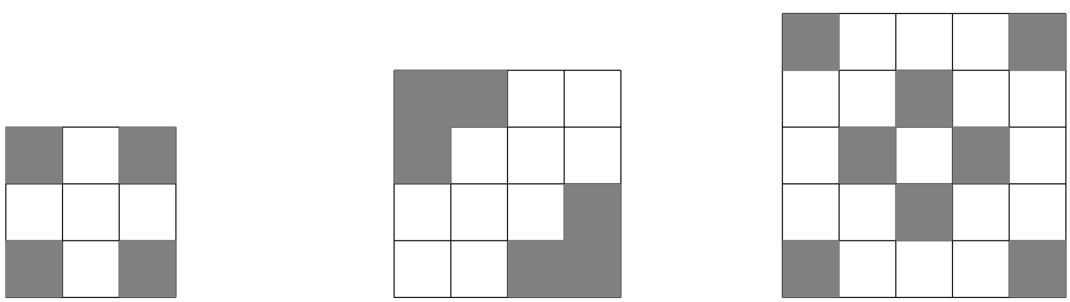
Figure 3. Examples of S_2-sufficient colorings for the cases $n = 4$.

Given that multiple possible colorings may exist, we will prove a method for producing additional X-sufficient colorings given two differing ones.

Proposition 5. *If C_1 and C_2 are X-sufficient colorings, then the coloring $C : c(x, y) = c_1(x, y) \oplus c_2(x, y)$ is X-sufficient (where $\oplus$ denotes the* `XOR` *operator).*

Proof. Assume coloring C_1 is X-sufficient. For a given cell (x, y) with color $c_1(x, y)$, we notice that its color in C remains the same if $c_2(x, y) = 0$ and changes if $c_2(x, y) = 1$. Any placement of the polyomino X will contain an even number of squares such that $c_2(x, y) = 1$, since C_2 is also X-sufficient. As X originally covered an even number of squares and has an even number of squares that change color, it still covers an even number of black squares. ■

The W-pentomino. We now consider the existence of sufficient colorings for the W-pentomino (Figure 1). By inspection, 3×3, 4×4, and 5×5 grids are W-sufficient. W-sufficient colorings for each grid are shown in Figure 4.



(a) $n = 3$ **(b)** $n = 4$ **(c)** $n = 5$

Figure 4. Examples of W-sufficient colorings for the cases $n = 3, 4, 5$.

The following lemma proves that the coloration of certain cells propagates throughout an $n \times n$ square grid.

Lemma 6. *If a coloring C is W-sufficient and $\{|x_1 - x_2|, |y_1 - y_2|\} = \{2, 3\}$, then $c(x_1, y_1) = c(x_2, y_2)$.*

Proof. Suppose a coloring C is W-sufficient. We will show that $c(x, y) = c(x + 3, y + 2)$ since other equalities can be shown analogously. The sums

$$\begin{aligned} &c(x, y) + c(x + 1, y) + c(x + 1, y + 1) + c(x + 2, y + 1) + c(x + 2, y + 2) \\ \equiv &c(x + 1, y) + c(x + 1, y + 1) + c(x + 2, y + 1) + c(x + 2, y + 2) + c(x + 3, y + 2) \pmod 2 \end{aligned}$$

since they count the number of black cells in two overlapping pentominos (see Figure 5). Therefore, their difference $c(x+3, y+2) - c(x, y) \equiv 0 \pmod 2$. Since $c(x, y)$ can only take a value of 0 or 1, the absolute value of the difference $|c(x+3, y+2) - c(x, y)|$ is both even and less than 2, so it must be equal to 0, or

$$c(x+3, y+2) - c(x, y) = 0 \implies c(x, y) = c(x+3, y+2).$$

■

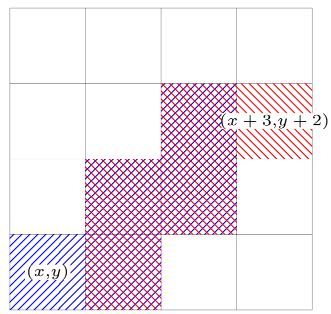


Figure 5. Two overlapping W-pentominos shown in blue and red demonstrate why $c(x, y) = c(x+3, y+2)$.

After showing this, we can provide a solution to the initially proposed problem from the Mathematics Magazine [1] previously outlined in the abstract.

Proposition 7. *An $n \times n$ grid is W-insufficient for $n \geq 6$.*

Proof. By Corollary 2, we must only show that a 6×6 grid is W-insufficient.

Suppose a W-sufficient coloring C exists. Applying Lemma 6, we see that $c(1, 1) = c(4, 3)$. By repeatedly applying Lemma 6 using the code provided in the Appendix, we can see that $c(1, 1) = c(x, y)$ for all cells (x, y) in the grid. This repeated procedure is depicted in Figure 6. Thus, in order for C to be a W-sufficient coloring, all cells must have the same color. However, this fails the requirement for both colors must be present. ■

10	3	6	5	8	3
3	8	7	4	5	8
6	7	2	9	4	5
5	4	9	2	7	6
8	5	4	7	9	3
1	8	5	6	3	10

Figure 6. The steps of the algorithm at which we know that the cell must be the same color as $(1, 1)$, as progressively inferred by Lemma 6.

Remark 1. The overlaying property proven in Lemma 6 can be extended to consider a number of other polyominos, where initial colorings propagate throughout a square grid.

The $(k\times1)$ polyomino and non-overlapping constructions. We present a generalization for $(k\times1)$ polyominos.

Theorem 8. *All $n \times n$ grids with $n > 1$ are sufficient for the $(k \times 1)$ polyomino*[1] *where $k > 2$.*

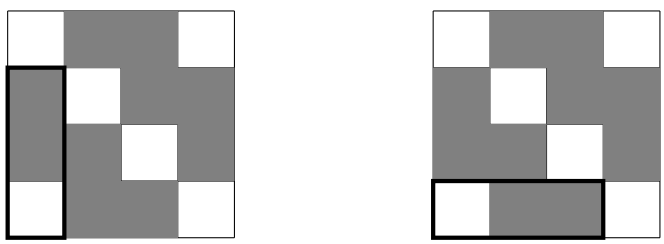
[1]And thus, by rotational symmetry, for $(1 \times k)$.

Proof. For $n < k$, the statement is vacuously true. We will construct a sufficient coloring for a $k \times 1$ polyomino. Color all cells (x, y) such that $x + y \equiv 0 \pmod{k}$ or $x + y \equiv 1 \pmod{k}$ black, and leave the others white. We will consider two cases.

- **The** $(k \times 1)$ **polyomino is placed vertically:** Let the coordinate of the bottom cell of the polyomino be (x_0, y_0), and note that $n - y_0 \geq 0$ as we assume the polyomino fits in the grid. Then, the polyomino covers all cells between (x_0, y_0) and $(x_0, y_0 + k - 1)$ inclusive, or the set of cells $\{(x_0, y_0 + a) : a \in [0, k-1], a \in \mathbb{Z}\}$ (From the fact that $n - y_0 \geq 0$, we know that these cells are contained in the grid). Recall that we have colored the cells such that $x + y \equiv 0 \pmod{k}$ and $x + y \equiv 1 \pmod{k}$ black. As x_0, y_0 are constant and a takes all integer values from 0 to $k - 1$ inclusive, we can see that there must be exactly two covered cells colored black.
- **The** $(k \times 1)$ **polyomino is placed horizontally:** Since the grid coloring is symmetric by reflection across the (i, i) diagonal, this case follows from the first.

As there are only two distinct rotations of the $(k \times 1)$ polyomino, we are done. ■

Example. Below are two example placements of a (3×1)-triomino on S_4 with the (3×1)-sufficient coloring described above.



(a) Non-rotated (3×1)-triomino in S_4 **(b)** Rotated (3×1)-triomino in S_4

Figure 7

Constructing polyominos from non-overlapping $(k \times 1)$ polyominos yields a powerful result.

Corollary 9. *Let X be a polyomino constructible by joining contiguous, non-overlapping $(1 \times k)$ and $(k \times 1)$ polyominos with $k > 2$. Then, the $n \times n$ square grid S_n is X-sufficient for all $n > 1$.*

Proof. Let $X = x_1 \cup x_2 \cup \ldots \cup x_l$ be a composite polyomino, where each x_i is a $1 \times k$ or $k \times 1$ polyomino defined by a set of ordered pairs for some $k > 2$, and $x_i \cap x_j = \emptyset$ when $i \neq j$. Color the grid according to the rule defined in Theorem 8 so that, by the theorem, each x_i covers an even number of black squares. Since $x_i \cap x_j = \emptyset$ for $i \neq j$, the total number of black squares covered by X is the sum of those covered by each x_i, which is necessarily even. This yields the result[2]. ■

This result is particularly powerful when considering how many different types of polyominos can now be shown to be X-sufficient.

Example. Below is an example of a composite hexomino consisting of two (3×1)-triominos.

[2]Note: If X does not fit on the square grid, the corollary is vacuously true; in general we exclude $n = 1$ as it cannot fulfill the condition of both colors being present for X-sufficiency.

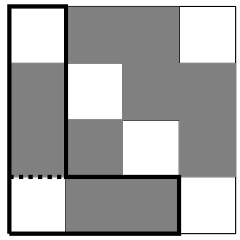


Figure 8. A hexomino composed of a (3×1)- and (1×3)-triomino on S_4. Thus S_4 is X-sufficient.

Additionally, Corollary 9 allows us to extend our consideration to a rectangular polyomino of arbitrary dimensions.

Proposition 10. *For any $a \times b$ rectangular polyomino R with $\max(a, b) > 2$, the $n \times n$ grid is R-sufficient for all $n > 1$.*

Proof. Without loss of generality, suppose that $a \geq b$. Then R can be formed by joining b copies of $(1 \times a)$ polyominos, and so by Corollary 9, the $n \times n$ grid is R-sufficient. ∎

The O-octomino. We briefly consider the O-octomino, shown in Figure 9.

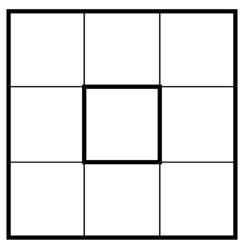


Figure 9. The O-octomino

Proposition 11. *All $n \times n$ grids are O-sufficient.*

Proof. For $n < 3$, the O-octomino does not fit, so the statement is vacuously true.

Label the $n \times n$ grid with $n \geq 3$ as usual and color squares with x- and y- coordinates of the same parity black, and leaving the others white. Now consider the octomino composed of a 3×3 square with the center removed. We will proceed with two cases.

Case 1: The O-octomino is centered at coordinates of differing parity. Without loss of generality, suppose that it is centered on a cell with an even x and odd y (the reverse is the same). Consider the diagram below:

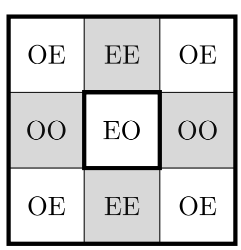


Figure 10. The O-octomino centered at coordinates of differing parity.

As we can see in Figure 10, exactly four black squares are covered in this case.

Case 2: The O-octomino is centered at coordinates of the same parity. Without loss of generality, suppose the polyomino is centered on a cell with both coordinates even. Consider the diagram below.

OO	EO	OO
OE	EE	OE
EO	OO	EO

Figure 11. The O-octomino centered at coordinates of the same parity. Note that the O-octomino does not contain the central cell.

We can again see in Figure 11 that exactly four black squares are covered.

Therefore, we see that the O-octomino always contains exactly four black cells with the given coloring, and so the proposition is true. ■

Odd X-sufficiency on triangular grids

We consider an extension of the initial problem to polyominos covering an odd number of colored cells instead of an even number. In this section, we will consider odd X-sufficiency on triangular grids in particular. We define the following.

Definition 6. (Odd X-sufficient) A grid is odd X-sufficient if there exists a coloring such that any placement and orientation of an X-polyomino covers an odd number of black cells.

We can immediately make determinations about odd-sufficiency for any polyomino X composed of an odd number of cells given even-sufficiency.

Proposition 12. *For some polyomino X that covers an odd number of cells, a grid that is X-sufficient is also odd X-sufficient.*

Proof. Since the grid is X-sufficient, we know that for some coloring, any placement and orientation of the X-polyomino covers an even number of black cells. However, since the polyomino covers an odd number of cells in total, it must cover an odd number of white squares. So, we can simply flip the colors of our X-sufficient coloring. Since the polyomino must now cover an even number of white squares, it must cover an odd number of black squares. ■

Triangular grids. We begin by defining a triangular grid.

Definition 7. (Triangular grid) We define T_n to be the equilateral triangular grid with a side length of n units.

Example. Figure 12 shows an example of the triangular grid T_n for $n = 5$.

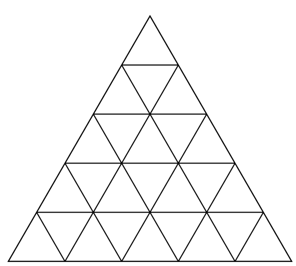
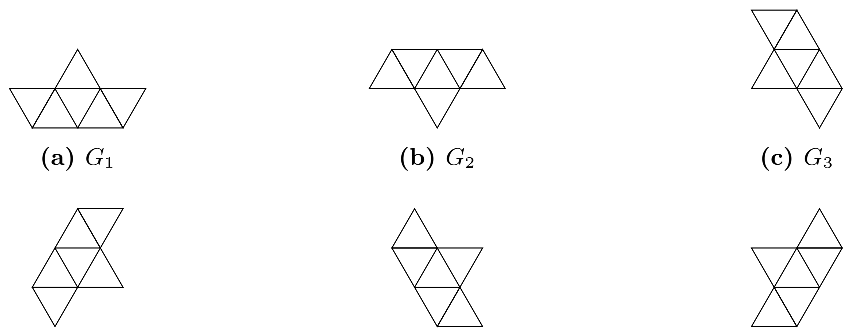


Figure 12. T_5.

The G-hexomino. We present a result concerning odd X-sufficiency on the triangular grid when X is the "Georgie" or G-hexomino, shown below in its six orientations.



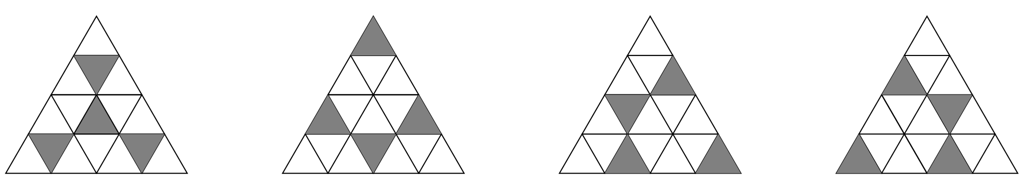
(a) G_1 **(b)** G_2 **(c)** G_3

(d) G_4 **(e)** G_5 **(f)** G_6

Figure 13. All six orientations of the G-hexomino placeable in a triangular grid T_n for $n \geq 4$.

Theorem 13. *T_n is odd G-sufficient for all $n \geq 4$.*

Proof. We first show that there exists a coloring such that any placement of G_1 not rotated covers an odd number of black cells. Consider $n = 4$. We use the following four colorings of T_4, which we refer to as "tiles," shown in Figure 14. Notice that there is only one valid placement of a non-rotated G_1 in each tile, and the polyomino covers an odd number of black cells in all cases.



(a) Tile A **(b)** Tile B_1 **(c)** Tile B_2 **(d)** Tile B_3

Figure 14. Valid odd colorings of T_4 for G_1. Tile A is unique, and tiles B_1, B_2, and B_3 are rotations of each other.

These will be the building blocks to construct T_n. We can produce a coloring of the remaining T_n grids by interlacing copies of A and each B_i perfectly. By construction, any T_4 subgrid of the new triangular grid is either tile A or one of B_1, B_2, B_3. Then, since each tile and therefore T_4 subgrid satisfies the property that any placement of G_1 without rotation covers an odd number of black cells (and G_1 can be fully contained in any T_4 subgrid), the entire grid would as well. This is shown in Figure 15.

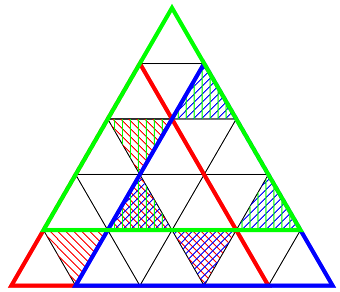


Figure 15. A T_5 grid can be formed by composing tiles A, B_1, and B_2, where A is shaded red, B_1 is shaded blue, and B_2 is shaded green.

Figure 16 shows a constructive example of T_8 from tiles A, B_1, B_2, and B_3.

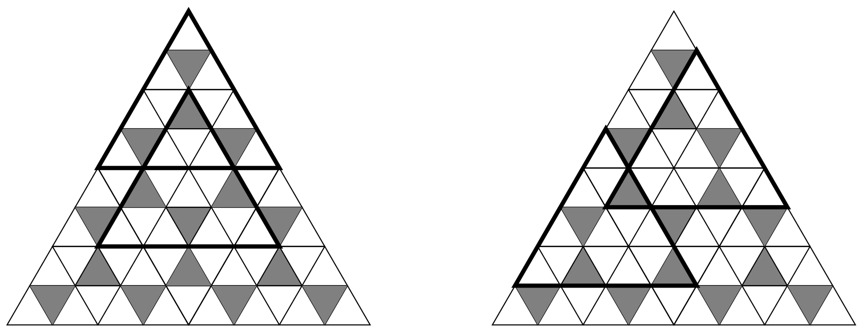


Figure 16. Valid odd coloring of T_8 for G_1. Notice that this triangular grid contains all tiles in Figure 14.

Therefore, for every $n \geq 4$, a coloring can be constructed by arranging or composing A and B_i tiles such that all placements of G_1 without rotation cover an odd number of black cells. Since each construction preserves this property, it holds true on T_n for any $n \geq 4$. Through this tiling method, we can construct a G_1-sufficient coloring of any given triangular grid. Since our plane coloring is identical under the same rotations by which $G_2, G_3 \ldots, G_6$ derive from G_1, we can determine that the property holds true for all orientations of G_1, and thus the coloring is odd G-sufficient. ■

Future Work

It may be interesting to explore additional grids of various structures as done with the triangular grid. We present a theorem and potential applications that would likely be useful in doing so.

Theorem 14. *Let T_Δ be a polyomino in a grid δ and ϕ be a bijective mapping from δ to a grid χ. Then, δ is T_Δ-sufficient iff there exists a coloring $\mathcal{C}$ of χ such that $\phi(T_\Delta)$ covers an even number of black cells in χ for all orientations and placements of T_Δ in δ.*

Proof. First, we will show that if δ is T_Δ-sufficient, then there is a coloring $\mathcal{C}$ of χ such that $\phi(T_\Delta)$ covers an even number of black cells in χ for all orientations and placements of T_Δ in δ. Assume δ is T_Δ-sufficient, so there exists a coloring $\mathcal{C}$ of δ such that any placement and orientation of T_Δ in δ covers an even number of black cells. Color χ by the rule that $\phi(x, y) \in \chi$ is black iff $(x, y) \in \delta$ is black.

Now, suppose that for some placement of T_Δ in δ, it is true that $\phi(T_\Delta)$ does not cover an even amount of black cells in χ colored with the above rule. Apply ϕ^{-1} to $\phi(T_\Delta)$ (We know this mapping exists as ϕ is bijective), yielding T_Δ. Because of the rule by which we colored χ, it must be true that T_Δ now does not cover an even number of black cells in δ with the coloring $\mathcal{C}$. However, this violates our assumption about the nature of the coloring $\mathcal{C}$, and so we have reached a contradiction.

For the other direction, suppose that there exists a coloring $\mathcal{C}$ of χ such that $\phi(T_\Delta)$ covers an even number of black cells in χ for all orientations and placements of T_Δ in δ, but that δ is T_Δ-insufficient. Color δ such that a cell $(x, y) \in \delta$ is black iff $\phi(x, y) \in \chi$ is black in $\mathcal{C}$. Now, suppose that for some placement of T_Δ in δ, it is true that T_Δ does not cover an even amount of black cells in δ colored under this rule. Apply ϕ to T_Δ. Because of the rule by which we colored δ and that ϕ is injective (as it is bijective), it must be true that $\phi(T_\Delta)$ now does not cover an even number of black cells in χ with the coloring $\mathcal{C}$. However, this violates our assumption about the nature of the coloring $\mathcal{C}$. So, we have again reached a contradiction. ■

Consider the grid T_5 as previously defined. We can map T_5 to a grid of squares using a bijection ϕ which, in combination with Theorem 14, will allow us to use our previous results for grids composed of squares. To illustrate this mapping, define the leftmost and bottom-most upward-pointing triangle in the triangle grid (resembling Δ) as $(1, 1)$. Define the triangles to the right by adding to the x coordinate and those above by adding to the y.

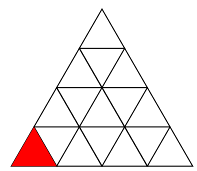


Figure 17. T_5 with $(1, 1)$ shown in red.

Let us define ϕ as mapping $(x, y) \in \delta \to (x, y) \in \chi$, where χ is a grid tiled by squares. We observe that isomorphic polyominos in δ are not always isomorphic after the mapping ϕ is applied, as shown in Figure 18.

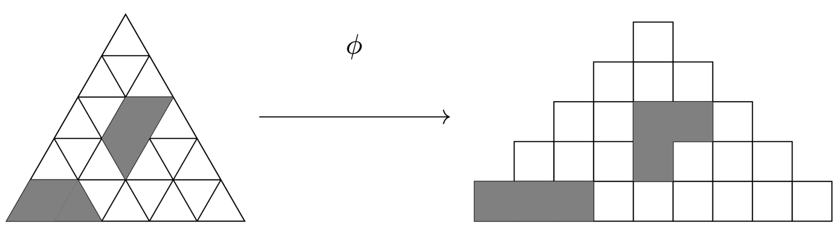


Figure 18. Mapping of T_5 containing two polyomoinos to a square grid under ϕ.

The triangular grid in Figure 18 contains two trapezoid triominos, or trapominos, composed of three triangular cells. Two rotations of this polyomino are shown below.

(a) Trapezoid triomino **(b)** Alternate orientation

Figure 19. Two orientations of the trapezoid triomino.

By inspection, we can determine that there is no $n > 2$ such that the grid T_n is trapomino-sufficient. We posit that this is because mapping a rotation of the trapomino in the triangle grid to the square grid gives the V-triomino (see Figure 18), for which we have established the common impossibility of creating a sufficient coloring.

From this we can gain another insight: rotating polyominos in grids with non-similar cells will change the adjacency of the cells that compose the polyomino after the mapping. It may also be interesting to examine X-semisufficient colorings, or colorings such that any placement without rotation of X on a grid covers an even number of black squares, as our results would be easier to use.

For another potential application of Theorem 14, consider a grid δ that maps to S_5 under the bijection $\phi : \delta \to S_5$, and suppose some polyomino X in δ maps to either of (but only to) the two polyominos shown below.

Figure 20. Two polyominos placeable in S_5.

Since we know that there exists a single coloring of S_5 that is sufficient for both of these polyominos (the checkerboard coloring), there must be an X-sufficient coloring of δ given by $c_\delta(x, y) = c_{S_5}(\phi(x, y))$.

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References

[1] Problems and Solutions. Problems and solutions. *Mathematics Magazine*, 96(3):359–369, 2023.

Appendix

Automated propagation of color equality (Python).

```
def check(x, y, n):
    return (0 <= x < n) and (0 <= y < n)

n = 6

a = [[0] * n for i in range(n)]
a[0][0] = 1

for t in range(10):
    for i in range(n):
        for j in range(n):
            for (dx, dy) in [(-2, -3), (-2, +3), (2, -3), (2, +3),
                    (+3, -2), (+3, +2), (-3, -2), (-3, +2)]:
                if (check(i + dx, j + dy, n)) and a[i][j]:
                    a[i + dx][j + dy] = 1

for i in range(n):
    print(a[i])
```