A BRIEF INTRODUCTION TO BÉZIER CURVES

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ABSTRACT. In this paper we will define the Bézier curve and introduce several interesting properties. We will also introduce the Bernstein polynomial and Bernstein basis polynomials, and state how they relate to De Casteljau's algorithm and rational Bézier curves. We will conclude with a short derivation of the derivative of the Bézier curve, alongside an outline for a method of calculating the derivative of a Bézier curve at a given point.

CONTENTS

INTRODUCTION

Bézier curves are discrete parametric curves that can be modeled and modified by points known as control points. From a theory-based perspective, Bézier curves are a special case of a category of curves known as B-spline curves, and can be composed with one another to form a complete B-spline curve. We will not discuss B-splines as they lie beyond the scope of this paper.

At first, Bézier curves were created to approximate smooth curves in real-world scenarios, and were incredibly useful in that regard, as they could easily be controlled using the aforementioned control points, and were easy to manipulate by a user. The namesake of the Bézier curve was a French engineer by the name of Pierre Bézier, who used it in the 1960s to innovate and model curvature on cars manufactured by Renault, a French car manufacturer. While Bézier curves are named after Bézier, there were numerous other contributors to its early development, especially with the French Paul de Casteljau, who developed an algorithm

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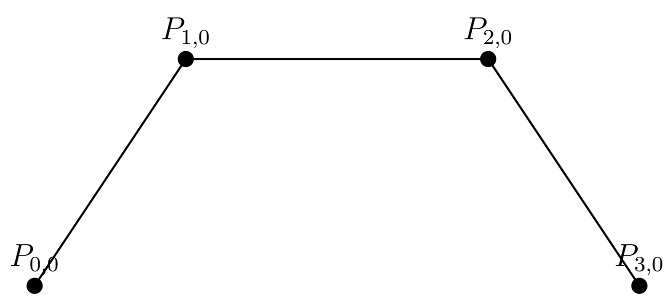
for computing Bézier curves, and the Ukrainian-Russian Sergei Natanovich Bernstein, who created the Berstein polynomial and Berstein basis polynomials, which can be used to represent Bézier curves.

Bézier curves have numerous applications in computer-aided design, animation, vector graphics, and many more.

For more information on cubic Bézier curves, [KA00] is a good place to begin, as it provides more insight on the calculation of Bézier curves using matrices. For more information on the different types of Bézier curves and uses in approximating segments of conic sections, [Rei11] is a good place to look. For far more insight on different components of Bézier curves, as well as more information on B-spline curves and other forms of continous smooth curves, [CK98] is a good start.

1. Creating the Bézier curve

To begin, let us consider 4 points. We will call points such as these **control points** throughout the remainder of this paper. Define these 4 points as $P_{0,0}$, $P_{1,0}$, $P_{2,0}$, and $P_{3,0}$. The second subscript indicates a characteristic known as the **level** of the curve, and the first subscript numbers the control point. Connect the control points to form a nearly complete trapezoid. The figure below will form the 0th level of our Bézier curve.



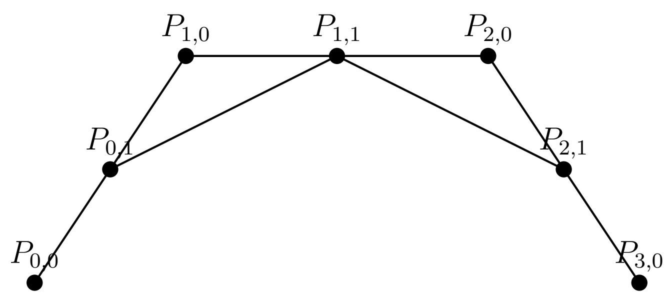
From this, we are able to define a technique known as linear interpolation. Define a parameter t. Let there be some point $P_{0,1}$ on the line segment $P_{0,0}P_{1,0}$. Let $t \in [0, 1]$. We may consider these control points to be vectors, and thus write an equation representing $P_{0,1}$ in terms of t.

$$P_{0,1} = tP_{0,0} + (1 - t)P_{1,0}$$

Notice that if we allow $t = 0$, then $P_{0,1} = P_{1,0}$, and if we allow $t = 1$, then $P_{0,1} = P_{0,0}$. Thus, by varying the value of t, we are changing the division of the segment $P_{0,0}P_{0,1}$. This process of parametrizing points on segments formed by these segments is known as **linear interpolation**. We can repeat this process for all other segments to obtain a list of equations that describe movement of points along the segments above.

$$P_{0,1} = tP_{0,0} + (1-t)P_{1,0}$$
$$P_{1,1} = tP_{1,0} + (1-t)P_{2,0}$$
$$P_{2,1} = tP_{2,0} + (1-t)P_{3,0}$$

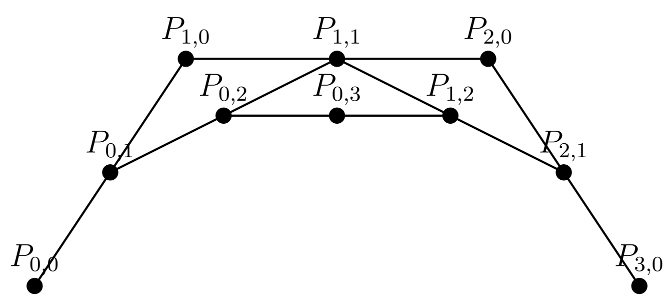
If we plot these points on the graph, we can observe the formation of a structure.



Finally, to complete the graph, define two more points $P_{0,2}$ and $P_{1,2}$ that lie on the segments $P_{0,1}P_{1,1}$ and $P_{1,1}P_{2,1}$ respectively, and add another point $P_{0,3}$ that moves along the segment $P_{0,2}P_{1,2}$. Using the same parameter t, we have

$$P_{0,2} = tP_{0,1} + (1-t)P_{1,1}$$
$$P_{1,2} = tP_{1,1} + (1-t)P_{2,1}$$
$$P_{0,3} = tP_{0,2} + (1-t)P_{1,2},$$

and when graphed, it looks like this:



What we have constructed above is a collection of control points that define a Bézier curve, where $t \in [0,1]$, and it is defined by

$$P_{0,1} = tP_{0,0} + (1-t)P_{1,0}$$
$$P_{1,1} = tP_{1,0} + (1-t)P_{2,0}$$
$$P_{2,1} = tP_{2,0} + (1-t)P_{3,0}$$
$$P_{0,2} = tP_{0,1} + (1-t)P_{1,1}$$
$$P_{1,2} = tP_{1,1} + (1-t)P_{2,1}$$
$$P_{0,3} = tP_{0,2} + (1-t)P_{1,2}.$$

Now that we have explicitly parametrized the curve, let us attempt to find an expression that will give us the Bézier curve in terms of t and all points. Noticing that $P_{0,3}$ can be rewritten with other expressions, write

$$\begin{aligned} P_{0,3} &= t(tP_{0,1} + (1-t)P_{1,1}) + (1-t)(tP_{1,1} + (1-t)P_{1,2}) \\ &= t(t(tP_{0,0} + (1-t)P_{1,0}) + (1-t)(tP_{1,0} + (1-t)P_{2,0})) \\ &+ (1-t)(t(tP_{1,0} + (1-t)P_{2,0})), \end{aligned}$$

and so on. As one may ascertain form the sheer scale of this expression, it becomes obvious that it would be pointless to do this manually, and a significant waste of time. Thus, it remains an exercise for the reader to prove that this yields

$$P_{0,3} = (1-t)^3 P_{0,0} + 3t(1-t)^2 P_{1,0} + 3t^2(1-t)P_{2,0} + t^3 P_{3,0}. \tag{1}$$

This curve is known as a **cubic Bézier curve**. Since the path of the point $P_{0,3}$ is easily determined by a smooth curve, we will denote $P_{0,3}$ as a function parametrized by t. We will denote the path formed by the function as $B_3(t)$, or more generally, $B_n(t)$. We refer to such polynomials as **Bernstein polynomials**. A characteristic of the Bézier curve is that it is infinitely differentiable and continous everywhere.

2. Higher-order Bézier curves

2.1. **Bernstein polynomials.** Of course, cubic Bézier curves are not the only Bézier curves that we can construct. What we constructed in section 1 was a Bézier curve of degree $n = 3$. However, we can also construct Bézier curves of degree $n = 1$, $n = 2$, $n = 5000$, and so on.

Remark 2.1. A Bézier curve of degree n must have $n + 1$ control points.

Naturally, a curve with $n = 1$ is simple linear interpolation along a straight line between two control points, and a curve with $n = 2$ is a quadratic Bézier curve with 3 levels of complication. If we observe the polynomial in (1), its coefficients seem oddly similar to binomial coefficients. This is no coincidence.

Definition 2.2 (Bernstein basis polynomial)**.** We define a Bernstein Basis Polynomial to be

$$b_{i,n}(t) = \binom{n}{i}(1-t)^{n-i}t^i,$$

where i denotes the control point number from 0 to n.

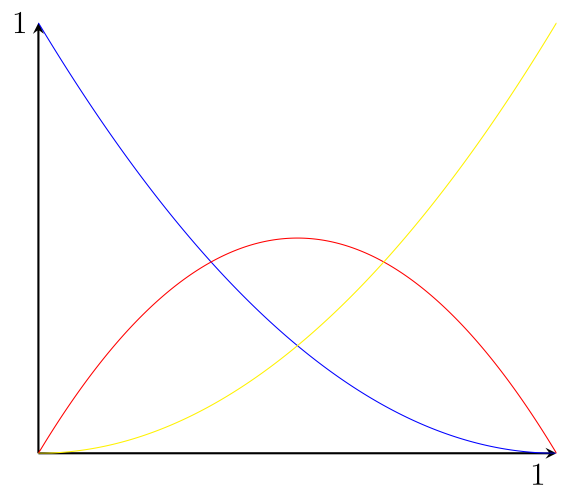
Definition 2.3 (Bernstein polynomial of degree n)**.** We define the Bernstein polynomial of degree n to be a linear combination of Berstein basis polynomials, or

$$\begin{aligned} B_{i,n}(t) &= \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i P_{i,0} \\ &= \binom{n}{0}(1-t)^n t^0 P_{0,0} + \binom{n}{1}(1-t)^{n-1} t^1 P_{1,0} + \cdots \\ &+ \binom{n}{n-1}(1-t)^1 t^{n-1} P_{n-1,0} + \binom{n}{n}(1-t)^0 t^n P_{n-1,0}. \end{aligned}$$

As an example, consider the Bernstein polynomial of degree $n = 2$. Then we have

$$\begin{aligned} B_{i,2} &= \binom{2}{0}(1-t)^2 P_{0,0} + \binom{2}{1}(1-t)tP_{1,0} + \binom{2}{2} t^2 P_{2,0} \\ &= (1-t)^2 P_{0,0} + 2t(1-t)P_{1,0} + t^2 P_{2,0}, \end{aligned}$$

with Bernstein basis polynomials $(1-t)^2$, $2t(1-t)$, and t^2. If we allow $t \in [0,1]$ and plot the individual Bernstein basis polynomials, then we can notice something very informative about the Bézier curve.



As we increase the value of t, the amount of weight applied to each control point is manipulated, so that the sum of the Bernstein basis polynomials evaluated certain points always equates to 1.

The same is true for Bézier curves of even higher degree. Consider a Bézier curve of degree $n = 7$. By 2.1, there must be 8 control points, namely $P_{0,0}$, $P_{1,0}$, $P_{2,0}$, $P_{3,0}$, $P_{4,0}$, $P_{5,0}$, $P_{6,0}$, and $P_{7,0}$. Then the Bernstein polynomial is

$$\begin{aligned} B_7(t) &= (1-t)^7 P_{0,0} + 7t(1-t)^6 P_{1,0} + 21t^2(1-t)^5 P_{2,0} + 35t^3(1-t)^4 P_{3,0} \\ &+ 35t^4(1-t)^3 P_{4,0} + 21t^5(1-t)^2 P_{5,0} + 7t^6(1-t)P_{6,0} + t^7 P_{7,0} \end{aligned}$$

with Bernstein basis polynomials

$$
\begin{aligned}
b_{0,7} &= (1-t)^7 \\
b_{1,7} &= 7t(1-t)^6 \\
b_{2,7} &= 21t^2(1-t)^5 \\
b_{3,7} &= 35t^3(1-t)^4 \\
b_{4,7} &= 35t^4(1-t)^3 \\
b_{5,7} &= 21t^5(1-t)^2 \\
b_{6,7} &= 7t^6(1-t) \\
b_{7,7} &= t^7.
\end{aligned}
$$

If we graph this, we obtain

Although this graph is far more complicated, it follows the same rules of a quadratic Bézier curve.

2.2. **De Casteljau's algorithm.** An efficient way to calculate Bézier curves is with De Casteljau's algorithm. As we showed in the first section, we construct a cubic Bézier curve by examining linear interpolation along each segment connecting each control point, and iterating the process until we reach a final point on the highest level. This process of calculating points can be generalized to Bézier curves of higher degree, and that is exactly what De Casteljau's algorithm does.

Consider some sequence of control points $P_{0,0}$, $P_{1,0}$, $P_{2,0}$,..., $P_{i,0}$. The number of levels on some Bézier curve is given by the degree of the particular curve. Let us denote the degree of the curve with n. Thus, after performing linear interpolation on the control points above, we will obtain a sequence of points $P_{0,1}$, $P_{1,1}$, $P_{2,1}$,..., $P_{i-1,1}$. Iterating once more with these points, we will obtain a set of sequences of points, where each sequence decreases by 1 element in size per iteration until it reaches a cardinality of 1, or a level of n. Then we will have the set

$$\{\{P_{0,0}, P_{1,0}, P_{2,0}, \ldots, P_{i,0}\}, \ldots, \{P_{0,k}, P_{1,k}, P_{2,k}, \ldots, P_{i-k,k}\}, \ldots, \{P_{0,n}\}\}$$

for $1 \leq k \leq n$.

If we want to find some $P_{i,n}$ and represent its linear interpolation with a recurrence relation of other control points, we can use this expression:

$$P_{i,n} = tP_{i,n-1} + (1-t)P_{i+1,n-1},$$

where i runs over all points on that level, and n denotes the level of the point. We may also express this with a diagram:

$$\begin{array}{ccc} P_{i+1,n-1} & \xrightarrow{1-t} & P_{i,n} \\ & \overset{t}{\nearrow} & \\ P_{i,n-1}, & & \end{array}$$

The arrows illustrate a multiplicative relationship between the recursive points and the point $P_{i,n}$.

2.3. **Rational Bézier curves.** As shown with the Bézier curves above, variation and manipulation of control points and the shape of the curve can be incredibly useful and beautiful. However, if we would like a greater degree of variation and control over the resulting shape of the curve, we can use rational Bézier curves.

Definition 2.4. A rational Bézier curve is defined as

$$B(t) = \frac{\sum_{i=0}^{n} \binom{n}{i}(1-t)^{n-i}t^i P_{i,0} w_i}{\sum_{i=0}^{n} \binom{n}{i}(1-t)^{n-i}t^i w_i},$$

for control points $P_{0,0}$, $P_{1,0}$, $P_{2,0}$, ..., $P_{n,0}$, where w_i is a weight that is applied to each control point to further increase variation.

Notice that the numerator is a weighted Bézier curve whereas the denominator is a weighted sum of Bernstein basis polynomials. The effect that these weights can have on the curve vary, but $w_i \neq 0$.

To graph a rational Bézier curve in $\mathbb{R}^n$, we generally consider a projection of points from $\mathbb{R}^{n+1}$ - which we will refer to as the homogeneous plane containing homogeneous coordinates - to $\mathbb{R}^n$. For example, if we want to construct a Bézier curve in $\mathbb{R}^2$ then we have to consider a projection of points from $\mathbb{R}^3$ onto a plane, where $\mathbb{R}^3$ acts as the homogeneous plane.

We can also use Bézier curves to exactly generate segments of conic sections. This is largely possible due to the discrete nature of the control points and flexibility of the curve.

While these topics of further exploration are indeed very interesting, they lie beyond the scope of this paper.

3. Derivative of the Bézier curve

The last characteristic of the Bézier curve that we will discuss is its derivative and repeated derivatives. The first derivative of the Bézier curve can be useful when discussing the tangent and normal vectors of a Bézier curve. Let us compute the first derivative of a Bézier curve of degree n.

To begin, first compute the derivative of the Bernstein basis polynomial.

$$\begin{aligned}\frac{d}{dt}\binom{n}{i}(1-t)^{n-i}t^i &= \frac{n!}{i!(n-i)!}((-1)(n-i)(1-t)^{n-i-1}t^i + (1-t)^{n-i}it^{i-1}) \\ &= \frac{n!}{i!(n-i)!}((1-t)^{n-i}it^{i-1} - (n-i)(1-t)^{n-i-1}t^i) \\ &= \frac{n!}{(i-1)!(n-i)!}(1-t)^{n-i}t^{i-1} - \frac{n!}{i!(n-i-1)!}(1-t)^{n-i-1}t^i \\ &= n\left(\frac{(n-1)!}{(i-1)!(n-i)!}(1-t)^{n-i}t^{i-1} - \frac{(n-1)!}{i!(n-i-1)!}(1-t)^{n-i-1}t^i\right) \\ &= n(b_{i-1,n-1} - b_{i,n-1}).\end{aligned}$$

Returning to the Bernstein polynomial, we can utilize this expression to calculate its derivative.

$$\begin{aligned}\frac{d}{dt}B_{i,n}(t) &= \frac{d}{dt}\left(\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}t^iP_{i,0}\right) \\ &= P_{0,0}\frac{d}{dt}\binom{n}{0}(1-t)^n + P_{1,0}\frac{d}{dt}\binom{n}{1}(1-t)^{n-1}t + \cdots + P_{n,0}\frac{d}{dt}\binom{n}{n}t^n \\ &= nP_{1,0}(b_{0,n-1} - b_{1,n-1}) + nP_{2,0}(b_{1,n-1} - b_{2,n-1}) + \cdots \\ &+ nP_{n-1,0}(b_{n-2,n-1} - b_{n-1,n-1}) \\ &= n\sum_{i=0}^{n-1} b_{i,n-1}(P_{i+1,0} - P_{i,0}) \\ &= n\sum_{i=0}^{n-1}\binom{n-1}{i}(1-t)^{n-i-1}t^i(P_{i+1,0} - P_{i,0}) \qquad (2)\end{aligned}$$

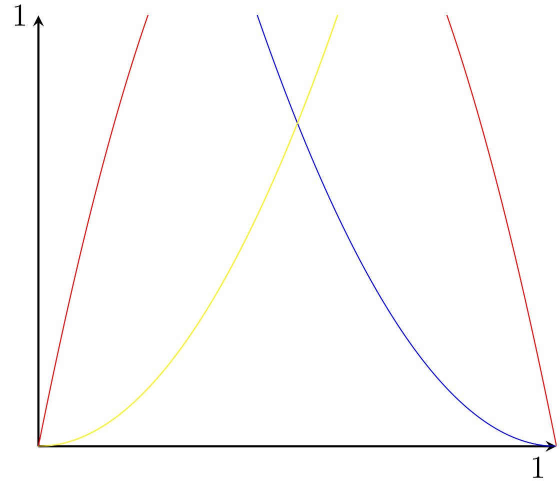
As an example, let us calculate the Bernstein polynomial for the derivative of a cubic Bézier curve.

$$\begin{aligned}\frac{d}{dt}B_{i,3}(t) &= 3\sum_{i=0}^{2}\binom{2}{i}(1-t)^{2-i}t^i(P_{i+1,0} - P_{i,0}) \\ &= 3\binom{2}{0}(1-t)^2(P_{1,0} - P_{0,0}) + \binom{2}{1}(1-t)t(P_{2,0} - P_{1,0}) + \binom{2}{2}t^2(P_{3,0} - P_{2,0}) \\ &= 3(1-t)^2(P_{1,0} - P_{0,0}) + 6t(1-t)(P_{2,0} - P_{1,0}) + 3t^2(P_{3,0} - P_{2,0}).\end{aligned}$$

We can also take the derivatives of individual Bernstein basis polynomials and graph them with the same restriction of $t \in [0,1]$. Thus, for a Bézier curve with $n = 3$, we have

$$
\begin{aligned}
b_{0,3} &= 3(1-t)^2 \\
b_{1,3} &= 6t(1-t) \\
b_{2,3} &= 3t^2.
\end{aligned}
$$

If we graph these we obtain



Now we contrast this with the Bernstein basis polynomials of a cubic Bézier curve.

$$
\begin{aligned}
b_{0,3} &= (1-t)^3 \\
b_{0,3} &= 3(1-t)^2 \\
b_{0,3} &= 3(1-t)t^2 \\
b_{0,3} &= t^3
\end{aligned}
$$

After graphing, we have

The derivative itself has its own meaning. As may be noticed from the diagrams in section 1, the interpolation of $P_{0,3}$ along the segment $P_{0,2}P_{1,2}$ is always tangent to the Bézier curve. The derivatives of the Bernstein basis polynomials also illustrate several intuitive characteristics of the cubic Bézier curve, and we can infer the nature of the cubic Bézier curve from its shape.

If we represent the sequence of control points with $Q_0 = n(P_{1,0} - P_{0,0})$, $Q_1 = n(P_{2,0} - P_{1,0})$,. . ., $Q_i = n(P_{i+1,0} - P_{i,0})$, we can rewrite the expression in (2) as

$$\frac{d}{dt}B_{i,n}(t) = \sum_{i=0}^{n-1} \binom{n-1}{i} (1-t)^{n-i-1} t^i Q_i.$$

As can be illustrated by the cubic Bézier curve example from above, the derivative of a Bézier of degree n is of one degree lower, and there are exactly n control points. We refer to the derivative of a Bézier curve as its **hodograph**.

If we want to calculate the derivative of the Bézier curve at an arbitrary point $t \in [0, 1]$, then we can utilize a relationship between De Casteljau's algorithm and the derivative. Notice that if we expand (2) in terms of its sums, then we obtain two individual Bézier curves $C_1(t)$ and $C_2(t)$.

$$\begin{aligned} n\sum_{i=0}^{n-1} \binom{n-1}{i} (1-t)^{n-i-1} t^i (P_{i+1,0} - P_{i,0}) &= n\left(\sum_{i=0}^{n-1} b_{i,n-1} P_{i+1,0} - \sum_{i=0}^{n-1} b_{i,n-1} P_{i,0}\right) \\ &= n(C_1(t) - C_2(t)). \end{aligned}$$

We will not outline it here, but it is possible to calculate $C_1(t) - C_2(t)$ using De Casteljau's algorithm. If we are able to, we can determine the exact value of the derivative of the Bézier curve at that point.

References

[CK98] Shene C.-K. *Introduction to Computing With Geometry Notes*. 1998.
[KA00] Shavez Kaleem and Al Abrahamsen. *Cubic Bézier Curves*. 2000.
[Rei11] Detlef Reimers. *Bézier Curves: An Introduction*. 2011.