# Q-ANALOGUES

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Abstract. In this expository paper, we will outline several interesting properties of the $q$-factorial and its relationship with inversions, the $q$-derivative and the $q$-analogue of Taylor's theorem, a theorem about the uniqueness of $q$-antiderivatives up to a constant alongside evaluation of definite $q$-integrals, as well as several perspectives and proofs of the $q$-binomial theorem. We will conclude with an introduction to the $q$-Pochhammer symbol, and use it to define the $q$-hypergeometric function and prove an interesting fact about the special $q$-hypergeometric series ${}_1\Phi_0$.

## Contents



## Introduction

The study of $q$-analogues, or $q$-theory, is a very diverse and somewhat new area of mathematics, and appears in numerous places such as combinatorics (especially in regard to the $q$-analogues of numerous combinatorial identities and properties, several of which we will outline here), number theory (we will not show this here, but it can be shown that the Euler function, an important analytic function in number theory and analysis, is a special case of the $q$-analogue of the Pochhammer symbol), and elsewhere. Specifically, the study of $q$-calculus, which we cover briefly in this paper, otherwise known as "quantum calculus", has long been a rather theoretical and playful tool, but in the past few decades, it has seen more frequent use in physics and areas of quantum mechanics. For an incredibly comprehensive overview of $q$-calculus and more details related to what is in this paper, [KC12] is an excellent reference. In the final section of this paper we will very briefly explore the notion of the

*Date*: Summer 2023.

hypergeometric function and its $q$-analogue. Though this topic seems disconnected, many techniques and lemmas from combinatorial $q$-theory are useful in deriving numerous special hypergeometric functions identities. In fact, Srinivasa Ramanujan considered several such special hypergeometric function identities and proved numerous remarkable results regarding them. We will not be exploring any special function identities nor their $q$-analogues, but more can be read about them in [GR04].

Though this paper covers the essentials of $q$-theory on an elementary level, modern $q$-theory is far more extensive and there are a variety of open problems. However, though it seems reasonable that any statement or expression can be expressed as a $q$-analogue, it is important to note that mathematicians are largely only interested in "useful" $q$-analogue generalisations; in other words, results that might be useful when generalised to a $q$-analogue. An example of this can be observed by the generalisation of the binomial coefficient to the $q$-binomial coefficient: it is not only incredibly useful, but forms a pillar for much of $q$-theory.

The $q$-analogue of some statement or expression is a parameterisation of that statement or expression in terms of some new parameter $q$, which, when equal to 1, yields the original statement. If the statement is undefined or the statement involves discontinuities when $q = 1$, then we can determine the $q$-analogue when $q$ approaches 1, i.e. when we consider $\lim_{q\to 1}$. When we refer to the $q$-analogue of some mathematical object, we often notate it with a subscript $q$. For instance, to denote the $q$-analogue of the natural numbers $n$, we write $[n]_q$. As we will be discussing the concept of hypergeometric series later, we will utilise the notation for falling and rising factorials. The falling factorial is the conventional factorial with descending terms and is denoted as $(x)_n$, whereas the rising factorial is denoted as $x^{(n)}$, is defined as follows.

**Definition 1** (Rising factorial or Pochhammer symbol)**.** We define the rising factorial as

$$x^{(n)} = x(x+1)(x+2)\cdots(x+k-1) = \prod_{k=1}^{n-1}(x+k).$$

The rising factorial is also known as the Pochhammer symbol. In this paper, we will be using it to denote the rising factorial, and will be using the notation $(x)_n$ for the sake of simplicity.

## 1. $q$-numbers and $q$-factorials

We will begin by introducing the $q$-analogue of the natural numbers $\mathbb{N}$.

**Definition 2** ($q$-analogue of the natural numbers)**.** We define

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \sum_{k=0}^{n-1} q^k.$$

We can see that this is consistent with our previous description because when $q = 1$, we produce the sum $\sum_k 1$, which generates the natural numbers. With this definition, we can express individual natural numbers in terms of polynomials with respect to the parameter $q$. However, before we can do this, notice that the RHS in the definition above is a finite geometric series. Since it is a finite geometric series, we can evaluate it as

$$\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q}.$$

The $q$-analogue of the integer 2, for example, can be expressed as

$$[2]_q = 1 + q = \frac{1-q^2}{1-q}.$$

Similarly, the $q$-analogue of the integer 10 can be expressed as

$$[10]_q = 1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + q^7 + q^8 + q^9 = \frac{1-q^{10}}{1-q}.$$

Recall that permutations are defined as the number of unique arrangements of $n$ objects, and can be evaluated as $n!$. Since we developed the notion of natural numbers above, we can thus define the $q$-analogue of the factorial, otherwise known as the $q$-factorial.

**Definition 3** ($q$-factorial)**.** We define the $q$-factorial as

$$\begin{aligned}[n]_q! &= \frac{1-q^n}{1-q} \cdot \frac{1-q^{n-1}}{1-q} \cdots \frac{1-q}{1-q} \\ &= \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)^n} \\ &= (1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2})\cdots(1+q).\end{aligned}$$

Let us informally introduce inversions. Given a sequence of numbers, an **inversion** is a swapping of two consecutive numbers into a different configuration. For example, if we have the sequence of integers 123, then the set of 1-inversions of 123 is given as $\{213, 132\}$. Similarly, the set of 0-inversions is $\{123\}$, the set of 2-inversions is $\{231, 312\}$, and the set of 3-inversions is $\{321\}$. Notice that the cardinality of the union of all inversion sets yields the total number of permutations of the sequence of integers. We write this as $3! = 6$. Say we want to write out the $q$-analogue of the factorial for $n = 3$. Then we have the expression

$$\begin{aligned}[3]_q! &= \frac{1-q^3}{1-q} \cdot \frac{1-q^2}{1-q} \cdot \frac{1-q}{1-q} \\ &= (1+q+q^2)(1+q) \\ &= 1 + 2q + 2q^2 + q^3.\end{aligned}$$

You may notice the emergence of a pattern, and this is no coincidence. There are 4 different inversions that we can perform on the canonical sequence 123, and there are 4 terms in the $q$-analogue form of 3!. Also, the coefficients of the polynomial for $[3]_q!$ are exactly the cardinality of each inversion set in order. This can be repeated for inversions of 1234, 12345, and so on, with the permutation count being $[4]_q!$ and $[5]_q!$. We can then conjecture that the number of monomials in the $q$-factorial is the exact number of inversions.

Let's introduce these notions more formally.

**Definition 4** (Inversion)**.** Let $w = w_1 w_2 \cdots w_n$ be a permutation of $S_n$, which is the group of all permutations of the sequence $w$. Let $w_k$ denote the number of numbers to the left of the $k$th number in the permutation. We define the inversion for $w$ to be a pair of numbers within the permutation $(w_i, w_j)$ such that $i < j$ but $w_i > w_j$.

For example, if we have the sequence 123, then $w_1 = 1, w_2 = 2, w_3 = 3$. If we want to conduct an inversion, we take a pair, say $w_1$ and $w_2$, and swap their positions such that

$1 < 2$ but $w_1 > w_2$, so that $w_2$ is 1 to the right from $w_1$. This permutation would result in 213, which is a 1-inversion. Now we can formalise the above conjecture.

**Proposition 1** (q-analogue of the factorial with inversion). *Let $a(m,n)$ denote the number of permutations of the sequence $1,2,\dots,n$ with exactly $m$ inversions. Then*

$$\sum_{m=0}^{\frac{n(n-1)}{2}} a(m,n)q^m = [n]_q!.$$

We will provide some intuition for our proof that will soon follow. Consider some unique configuration of numbers $\chi \in S_n$. From these unique configurations, we can make $n+1$ permutations $\chi_0, \chi_1, \dots, \chi_n \in S_{n+1}$. Since $\chi$ is a random configuration, we will consider instances in which we put the $n+1$th number into the individual $\chi_i$s, where $0 \leq i \leq n+1$. There are exactly $n+1$ ways to insert the $n+1$th number, and we do this by beginning from the right end and performing inversions with consecutive numbers until we reach the left end. If we continue as such, then the number of inversions as we increment the position of the $n+1$th number from right to left also increases by 1. So, if we begin from the right end, we have a 0-inversion; moving 1 to the left, we have a 2-inversion; and finally, moving $n$ to the left, we have an $n+1$-inversion.

*Proof of Proposition 1.* This will be a proof by induction on $n$. The case when the permutation consists of one number is trivial because there can only be no inversions on a permutation with 1 number. We assume that the property is true for $n$, and evaluate the property for $n+1$. Notice that since we are taking the $n+1$ permutation, we are inserting some $n+1$th number into the individual permutations of $n$ from $S_n$. Since the LHS of Proposition 1 is a generating function with coefficients of $q^0, q^1, \dots, q^n, q^{n+1}$, we can expand the sum

$$\begin{aligned}\sum_{m=0}^{\frac{n(n-1)}{2}} a(m,n+1)q^m &= 1\sum_{m=0}^{\frac{n(n-1)}{2}} a(m,n)q^m + q\sum_{m=0}^{\frac{n(n-1)}{2}} a(m,n)q^m + \cdots + q^n\sum_{m=0}^{\frac{n(n-1)}{2}} a(m,n)q^m \\ &= [n]_q!(1+q+q^2+\cdots+q^n).\end{aligned}$$

Notice from our original definition of the number $n$ that $[n] = 1+q+q^2+\cdots+q^{n-1}$, so that $1+q+q^2+\cdots+q^{n-1}+q^n = [n+1]$. Then

$$\sum_{m=0}^{\frac{n(n-1)}{2}} a(m,n+1)q^m = [n]_q![n+1]_q = [n+1]_q!.$$

By induction, this must be true for all permutations of $n$ in $S_n$, and we are done. ∎

## 2. $q$-binomial coefficients and the $q$-binomial theorem

2.1. **$q$-binomial coefficients.** Now that we have derived an interesting fact about the relationship between inversions and the $q$-factorial, let us introduce the notion of the Gaussian binomial coefficient, otherwise known as the $q$-analogue of the binomial coefficient. This will be useful in our proof of the $q$-binomial theorem in later sections.

**Definition 5** ($q$-analogue of the binomial coefficient)**.** We define

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!([n-k]_q!)}$$
$$= \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

To see that this is a $q$-analogue, notice that if we let $q = 1$, we have the standard definition of the binomial coefficient over $\mathbb{N}$. From this definition we can derive a number of combinatorial identities. An example of this can be seen through the $q$-analogue of Pascal's identity, which states that

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q.$$

A proof can follow by manipulating the definition of the $q$-binomial coefficient.

2.2. **The $q$-binomial theorem.** Now that we have the tools with which to define and prove the finite $q$-binomial theorem, we can follow as such.

**Theorem 2.1** (Finite $q$-analogue of the binomial theorem)**.**

$$(1+x)(1+qx)(1+q^2x)\cdots(1+q^{n-1}x) = \sum_{k=0}^{n}\binom{n}{k}_q q^{\frac{k(k-1)}{2}}x^k. \tag{2.1}$$

We can verify that this indeed is a $q$-analogue by letting $q = 1$, which reduces the LHS of the theorem to $(1+x)^n$. Then we are taking the standard binomial expansion of the binomial $(1+x)^n$. The proof that follows is a very nice proof by G. Polya and G.L. Alexanderson, and can be found in [PA71].

*Proof of Theorem 2.1.* Let us denote the LHS of 2.1 with the function $f(x)$. Observe that it is true that

$$(1+x)f(qx) = f(x)(1+q^nx) \tag{2.2}$$

because

$$(1+x)((1+qx)(1+q^2x)(1+q^3x)\cdots(1+q^nx)) = ((1+x)(1+qx)(1+q^2x)\cdots(1+q^{n-1}x))(1+q^nx).$$

Let us define the expansion of $f(x)$ as

$$f(x) = \sum_{k=0}^{n} Q_kx^k = Q_0 + Q_1x + Q_2x^2 + \cdots + Q_nx^n,$$

where the $Q_i$ denote the coefficients of the resulting polynomial expansion. There are several properties that are obvious, such as the fact that $Q_0 = 1$ and $Q_n = q^{0+1+2+\cdots+(n-1)} = q^{n(n-1)/2}$. Using 2.2, we can see that

$$(1+x)\sum_{k=0}^{n} Q_kq^kx^k = (1+q^nx)\sum_{k=0}^{n} Q_kq_kx^k.$$

We can rewrite this as

$$\begin{aligned}(1+x)(1+Q_1qx+Q_2q^2x^2+\cdots+Q_nq^nx^n) &= (1+q^nx)(1+Q_1x+Q_2x^2+\cdots+Q_nx^n)\\ Q_kq^k+Q_{k-1}q^{k-1} &= Q_k+q^nQ_{k-1}\\ Q_k(q^k-1) &= Q_{k-1}(q^n-q^{k-1})\\ Q_k &= Q_{k-1}\frac{q^{n-k+1}-1}{q^k}q^{k-1},\end{aligned}$$

where $k$ runs from 1 to $n$. Using the polynomial definition of the $q$-factorial, we can run through all possible coefficients $Q_k$ and will find that the $k$th term is in fact given by

$$Q_k=\binom{n}{k}_q q^{\frac{k(k-1)}{2}}.$$

Therefore, this equates to the LHS of 2.1, completing the proof. ■

## 3. $q$-CALCULUS

In this section we will develop some basic properties and analogies of $q$-calculus to classical calculus. Quantum calculus is divided into $q$-calculus, which is the $q$-analogue of calculus, and $h$-calculus, where $h=\hbar$ is Planck's constant. The $h$-derivative is defined by $\frac{\mathrm{d}_h f(x)}{\mathrm{d}_h x}=(f(x+h)-f(x))/h$. Notice how this is very similar to the classical definition of the derivative, but there is no $\lim_{h\to 0}$. The $h$-calculus is similar to the calculus of finite differences. In classical calculus, we frequently make use of the limit, especially in the definition of derivatives and integrals to make sense of approaching some value of incline and approximating area respectively. In $q$-calculus, however, we do not have the notion of limits.

### 3.1. Differential $q$-calculus.

**Definition 6** ($q$-derivative)**.** Let there exist some function $f$. Then the $q$-analogue of the derivative of $f$ in $x$ can be expressed as

$$\frac{\mathrm{d}_q f(x)}{\mathrm{d}_q x}=\lim_{q\to 1}\frac{f(qx)-f(x)}{qx-x}.$$

We remove the limit and say that $\frac{\mathrm{d}_q f(x)}{\mathrm{d}_q x}=\frac{f(qx)-f(x)}{qx-x}$. Notice that this definition conforms to the requirements for a $q$-analogue because when $q\to 1$, we have the same expression as for the classical derivative. From this we can see that the differential is $\mathrm{d}_q f=f(qx)-f(x)$. As an example, consider the $q$-analogue of the power rule. If we have some $x^n$, then

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x}(x^n)=\frac{x^n(q^n-1)}{x(q-1)}=\frac{q^n-1}{q-1}x^{n-1},$$

which is essentially analogous to the power rule in classical calculus as the expression $(q^n-1)/(q-1)$ is simply $[n]_q$. Recall the statement of Taylor's theorem in classical calculus. In order to derive the $q$-analogue of the theorem, we will have to derive a slightly different form of Taylor's theorem that has a linear-algebraic feel.

**Theorem 3.1** (Existence of alternative Taylor's theorem)**.** *Let $P_0,P_1,P_2,\dots,P_N$ be polynomials and $a\in\mathbb{R}$ be a number. Let these polynomials possess the properties that*

(1) $P_0(a)=1$ *and* $P_n(a)=0$ *when* $n\geq 1$,
(2) *deg*$(P_n)=n$,

(3) $\frac{d}{dx}P_n(x) = P_{n-1}(x)$ *when* $n \geq 1$.

*Then any polynomial* $f$ *with deg*$(f) = n$ *can be expressed as*

$$f(x) = \sum_{n=0}^{N} \frac{d^n f}{dx^n}\Big|_{x=a} P_n(x).$$

*Proof.* Let us define a vector space $V$ with polynomials of degree $\leq N$. Notice that the polynomials $P_i$ are linearly independent because their degrees are strictly increasing. Also notice that we can interpret the theorem as a linear combination of the polynomials $P_i$. Since these are true, the list of polynomials $P_0, P_1, P_2, \ldots, P_N$ must be a basis for $V$. Then we have

$$f(x) = \sum_{n=1}^{N} c_n P_n(x),$$

where $c_n$ denotes some coefficient that we are trying to determine. To verify that the form in the theorem is correct, let us differentiate $f(x)$ exactly $k$ times and evaluate at $a$. Notice that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}P_n(x) = P_{n-k}(x)$. Then

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x)\Big|_{x=a} = \sum_{n=1}^{N} c_n \frac{\mathrm{d}^k}{\mathrm{d}x^k} P_n(x)\Big|_{x=a} = \sum_{n=1}^{N} c_n P_{n-k}(a) = c_k.$$

Since we have shown that the coefficient is of this form, this is the same as Taylor's theorem, and we are done. ■

Notice that since the polynomials $P_0, P_1, \ldots, P_N$ form a basis for the vector space of polynomials $V$ with degree $\leq N$, the derivative, which we may denote as $D_q$ but will denote as the conventional derivative instead, acts as a linear operator in $V$. If we interpret the $q$-analogue in the same way, where the $q$-derivative acts as a linear operator on the vector space, then we have the same analogue. The statement and proof of the $q$-analogue of Taylor's theorem is identical.

**Corollary 1** (Existence of $q$-analogue of Taylor's theorem)**.** *Let the same conditions in Theorem 3.1 be true. Then any polynomial* $f$ *with deg*$(f) = n$ *can be expressed as*

$$f(x) = \sum_{n=0}^{N} \frac{d_q^n f}{d_q x^n}\Big|_{x=a} P_n(x).$$

While we have proven the existence of a $q$-analogue to Taylor's theorem, we have not found the polynomials $P_i$ that it describes. To find them it is necessary to derive the $q$-analogue of the product rule. However, before we do this, let us conjecture the structure of one such polynomial. By the conditions in Theorem 1, we must have that $P_0 = 1$. Then we must also have that $P_1 = x - a$. We can find, by using Corollary 1, that $P_2 = \frac{(x-a)(x-qa)}{[2]_q}$. If we continue to find the polynomials corresponding to $P_i$ with increasing $i$, we will find that a pattern emerges, so it seems that we have

$$P_n = \frac{\prod_{k=1}^{n}(x - q^k a)}{[n]_q!},$$

where $\prod_{k=1}^{n}(x - q^k a)$ is the $q$-analogue of the polynomial $(x-a)^n$, as can be seen by substituting $q = 1$. Notice how this bears a resemblance to the structure of a Taylor polynomial centered at $a$, but instead we are representing the numerator and denominator with their respective $q$-analogues. Note how this bears some resemblance to the $q$-Pochhammer symbol,

Let us now derive the product rule.

**Lemma 1** ($q$-analogue of the product rule)**.**

$$\frac{d_q}{d_q x}(f(x)g(x)) = f(qx)\frac{d_q}{d_q x}g(x) + g(x)\frac{d_q}{d_q x}f(x).$$

*Proof.* To begin, recall that we defined the differential of some function $u$ as $\mathrm{d}_q u = u(qx) - u(x)$. Then we have

$$\begin{aligned}
\mathrm{d}_q(f(x)g(x)) &= f(qx)g(qx) - f(x)g(x) \\
&= f(qx)g(qx) - f(qx)g(x) + f(qx)g(x) - f(x)g(x) \\
&= f(qx)(g(qx) - g(x)) + g(x)(f(qx) - f(x)) \\
&= f(qx)\mathrm{d}_q g(x) + g(x)\mathrm{d}_q f(x) \\
\frac{\mathrm{d}_q}{\mathrm{d}_q x}(f(x)g(x)) &= f(qx)\frac{\mathrm{d}_q}{\mathrm{d}_q x}g(x) + g(x)\frac{\mathrm{d}_q}{\mathrm{d}_q x}f(x).
\end{aligned}$$

■

Notice that there is indeed symmetry between $f$ and $g$, so we can interchange them to obtain

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x}(f(x)g(x)) = f(x)\frac{\mathrm{d}_q}{\mathrm{d}_q x}g(x) + g(qx)\frac{\mathrm{d}_q}{\mathrm{d}_q x}f(x). \tag{3.1}$$

Since the $q$-analogue of Taylor's theorem involves finding polynomials $P_i$, we need to verify that these $P_i$s indeed satisfy the requirements. The first and second conditions are trivially satisfied. The third condition would involve proving that we can take the derivative of $P_n$, or that we can take the derivative of the $q$-analogue of $(x-a)^n$; in other words, we want to show that by taking the derivative of $P_n$ of degree $n$, we obtain a polynomial $P_{n-1}$ that is of degree $n-1$. We are trying to show that $\frac{\mathrm{d}_q}{\mathrm{d}_q x}(x-a)_q^n = [n]_q(x-a)_q^{n-1}$. If we are able to show that this is true, then by repeated application we have shown that we can take the $j$th derivative, and that the third property holds.

*Proof that the $P_n$ satisfy the 3rd condition in Corollary 1.* The proof will be by induction on $k$. For the base case, it is trivial when $n = 1$. We assume that the statement is true for some integer $k$, so that $\frac{\mathrm{d}_q}{\mathrm{d}_q x}(x-a)_q^k = [k]_q(x-a)_q^n$. Then, by the definition of the $q$-analogue of $(x-a)^n$, we can write $(x-a)_q^{k+1} = (x-a)_q^k(x-q^k a)$, because the $q$-analogue of the $k+1$th factor is exactly $(x - q^k a)$. We then take the $q$-derivative, and using Lemma 1 as well as (3.1), we have

$$\begin{aligned}
\frac{\mathrm{d}_q}{\mathrm{d}_q x}\left((x-a)_q^k(x-q^k a)\right) &= (x-a)_q^k\frac{\mathrm{d}_q}{\mathrm{d}_q x}(x-q^k a) + (qx - q^k a)\frac{\mathrm{d}_q}{\mathrm{d}_q x}(x-a)_q^k \\
&= (x-a)_q^k + q(x-q^{k-1}a)[k]_q(x-a)_q^{k-1} \\
&= (x-a)_q^{k-1} + q[k]_q(x-a)_q^k \\
&= (1 + q[k]_q)(x-a)_q^{k-1} = [k+1]_q(x-a)_q^k = [k](x-a)_q^{k-1},
\end{aligned}$$

and by induction the statement is true. ■

The $q$-analogue of Taylor's theorem has many uses, and we will only outline one of its more important uses. Though [PA71] provides a beautiful proof of the $q$-analogue of the binomial theorem, the proof that follows from the $q$-analogue of Taylor's theorem is practical. Before we can prove it, we must formalise our results regarding the $q$-analogue of Taylor's theorem.

**Theorem 3.2** ($q$-analogue of the Taylor series expansion)**.** *Let $f(x)$ be a polynomial with $deg(f(x)) = N$, and let $a$ be a number. Then the $q$-Taylor series of $f(x)$ is*

$$f(x) = \sum_{k=0}^{N} \frac{d_q f(x)}{d_q x}\Big|_{x=a} \frac{(x-a)_q^k}{[k]_q!}.$$

This comes from our previous results. To see how this might prove the $q$-analogue of the binomial theorem, let us consider an example with the function $f(x) = x^n$ centered at $a = 1$. Let $k$ run from 0 to $\deg(f(x)) = n$. Then

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x} x^n = [n]_q[n-1]_q \cdots [n-k+1]_q x^{n-k}.$$

When evaluated at $a = 1$, we have

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x} x^n\Big|_{a=1} = [n]_q[n-1]_q \cdots [n-k+1]_q$$

Then the $q$-Taylor series for $x^n$ at $a = 1$ is

$$\sum_{k=0}^{n} \frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!}(x-1)_q^k = \sum_{k=0}^{n} \binom{n}{k}_q (x-1)_q^k,$$

since

$$\frac{[n]_q[n-1]_q \cdots [n-k+1]_q}{[k]_q!} = \binom{n}{k}_q.$$

The result (3.1) can be proven by considerirng the definition of the $q$-analogue of a natural number. With this intuition, we can prove the $q$-analogue of the binomial theorem differently.

*Proof of the $q$-analogue of the binomial theorem using the $q$-Taylor series.* We will be expanding the polynomial $f(x) = (x+a)_q^n$ about $x = 0$ using Theorem 3.2. We then have $\frac{\mathrm{d}_q}{\mathrm{d}_q x} x^n = [n]_q[n-1]_q \cdots [n-k+1]_q (x+a)_q^{n-k}$. Recall that for some $m$ we defined $(x+a)_q^m = (x+a)(x+qa)\cdots(x+q^{m-1}a)$. If $x = 0$, then we are left with $(a)(qa)\cdots(q^{m-1}) = a^m q^{0+1+\cdots+(m-1)} = a^m q^{\frac{m(m-1)}{2}}$. Substituting for $m = n-k$, we have

$$\frac{\mathrm{d}_q}{\mathrm{d}_q x} x^n\Big|_{x=0} = [n]_q[n-1]_q \cdots [n-k+1]_q q^{\frac{(n-k)(n-k-1)}{2}} a^{n-k},$$

so that the $q$-Taylor series is

$$(x+a)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{(n-k)(n-k-1)}{2}} a^{n-k} x^k.$$

This can be simplified by noticing that $\binom{n}{k}_q = \binom{n}{n-k}_q$ and substituting $n-k$ for $k$. Then

$$(x+a)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{n(n-1)}{2}} a^k x^{n-k},$$

which is the $q$-analogue of the binomial theorem. ■

Another important result related to $q$-calculus is Heine's binomial formula. This result is only possible if we define and show that any formal power series can be expressed as a generalised Taylor series, and as a corollary to this, a $q$-Taylor series centered at 0. To begin, let us define a formal power series.

**Definition 7** (Formal power series)**.** We say that a formal power series is an infinite series that does not contain a notion of convergence. In this way, it is a polynomial of infinite degree.

Since formal power series do not have a notion of convergence, it is possible to algebraically manipulate them and perform classical calculus on them. This brings us to the following theorem.

**Theorem 3.3** (Generalised Taylor's theorem for formal power series)**.** *Let* $\frac{d_q}{d_q x}$ *be a linear operator on the vector space of formal power series, and let the infinite list* $P_0(x), P_1(x), P_2(x) \ldots$ *satisfy the conditions for the polynomials* $P_i$ *described in Theorem 3.1 when centered at* $a = 0$*. Then any formal power series* $f(x)$ *on the vector space of formal power series can be expressed as a generalised Taylor series centered at* $x = 0$*.*

A corollary extended to $q$-analogues follows immediately.

**Corollary 2** ($q$-analogue of generalised Taylor's theorem for formal power series)**.** *Any formal power series* $f(x)$ *can be expressed as a* $q$*-Taylor series centered about* $x = 0$*.*

*Proof of Theorem 3.3.* We can see that, by induction on $n$, that the polynomial $P_n(x) = a_k x^k$, where $a_k$ is a nonzero coefficient. This is true because when we take successive derivatives and evaluate at $a = 0$, we produce a coefficient of $a_k$. Then for any formal power series $f(x)$, we must have

$$f(x) = \sum_{i=0}^{\infty} c_i P_i(x)$$

for some coefficient $c_i$. Since $f$ is a formal power series, we can apply the $q$-derivative linear operator $k$ times and evaluate at $x = 0$ to obtain that the $k$th coefficient is $c_k = \frac{\mathrm{d}_q^k}{\mathrm{d}_q^k x} f(x)\Big|_{x=0}$, which indeed shows that our formal power series

$$f(x) = \sum_{i=0}^{\infty} \frac{\mathrm{d}_q^k f(x)}{\mathrm{d}_q^k x}\Big|_{x=0} P_i(x),$$

and we are finished. ■

Since we have shown that we can expand any formal power series into a $q$-Taylor series, we can explicitly determine the $q$-Taylor series expansion of any formal power series. The formal power series in question is $1/(x-a)_q^n$. Notice that this is a formal power series by polynomial long division, and results in a polynomial of infinite degree.

**Proposition 2** (Heine's binomial formula)**.** *We have the following* $q$*-Taylor series expansion:*

$$\frac{1}{(x-a)_q^n} = \sum_{k=0}^{\infty} \frac{[n]_q [n+1]_q \cdots [n+k-1]_q}{[k]_q!} x^k. \tag{3.2}$$

Let us take the $q$-Taylor series expansion of the LHS. It is an exercise for the reader to show that the $q$-derivative of the LHS is $[n]_q/(1-x)_q^{n+1}$. This can be shown by deriving the $q$-quotient rule, which is a direct corollary to the $q$-product rule in Lemma 1. Continuing as such, we can see that the $k$th derivative evaluated at $x = 0$ will be

$$\begin{aligned}\frac{\mathrm{d}_q^k}{\mathrm{d}_q^k x}\left(\frac{1}{(x-a)_q^n}\right)\Big|_{x=0} &= \left(\frac{[n]_q[n+1]_q\cdots[n+k-1]_q}{(1-x)_q^{n+k}}\right)\Big|_{x=0}\\ &= [n]_q[n+1]_q\cdots[n+k-1]_q.\end{aligned}$$

Substituting this into the $q$-Taylor series, we obtain (3.2). This result will be useful later.

It is possible, using the $q$-analogue of Taylor's theorem and numerous other interesting facts, to derive a variety of other useful properties and mathematical objects, such as $q$-trigonometric functions, $q$-exponential functions, the $q$-Gamma and $q$-Beta functions, and so on. We will not cover these, but we will introduce the notion of integral $q$-calculus by defining and stating several properties of the $q$-antiderivative.

3.2. **Integral $q$-calculus.** The definition of the $q$-antiderivative is very intuitive.

**Definition 8** ($q$-analogue of the antiderivative)**.** Let $\frac{\mathrm{d}_q}{\mathrm{d}_q x}$ denote the $q$-derivative. Then the function $F(x)$ is a $q$-antiderivative of $f(x)$ if $\frac{\mathrm{d}_q}{\mathrm{d}_q x}F(x) = f(x)$. We denote this as

$$F(x) = \int f(x)\mathrm{d}_q x.$$

Notice that analogous to classical calculus, the $q$-antiderivative is unique up to a constant. In $q$-calculus, this constant need not be a numerical constant. If we let $\varphi(x)$ denote a constant term, then $\frac{\mathrm{d}_q}{\mathrm{d}_q x}\varphi(x) = 0$ if and only if $\varphi(qx) = \varphi(x)$, because letting $q = 1$ allows $\varphi(qx) = \varphi(x)$. However, $\varphi(x)$ need not be a constant term to satisfy this criterion, and can be a polynomial or any other expression involving $x$. Let $\varphi(x)$ be a formal power series. Then in the expression of a formal power series, we must have by the previous condition that there must exist some coefficient $c_n$ of $x^n$ such that $q^n c_n = c_n$. However, it is obvious that the only time this is true is when $c_n = 0$. Thus, if we let

$$\sum_{n=0}^{\infty} a_n x^n$$

be a formal power series, then, analogous to the integration of a power series or monomial in classical calculus, there exists some constant $C$ such that

$$\int\left(\sum_{n=0}^{\infty} a_n x^n\right)\mathrm{d}_q x = \sum_{n=0}^{\infty}\frac{a_n x^{n+1}}{[n+1]_q} + C.$$

This can be proven by induction on $n$. We consider the case when $\varphi(x)$ is not a formal power series, and is instead a general function.

**Theorem 3.4** (Uniqueness of the general $q$-antiderivative up to constant)**.** *Let $q \in (0,1)$. Then, up to adding a constant, any function $f(x)$ has at most one $q$-antiderivative that is continuous at $x = 0$.*

The proof that we will show uses techniques of analysis related to the infimum and supremum of a set. If we have some partially ordered set $P$ and a subset $S \subset P$, then we define the infimum to be the greatest $x \in P$ such that $x$ is less than or equal to every element of $S$, and we define the supremum to be the least $x \in P$ such that $x$ is greater than or equal to every element of $S$. In other words, the infimum of $P$ is the greatest element in $P$ such that it is less than or equal to $\min(S)$, and the supremum of $P$ is the smallest element in $P$ such that is is greater than or equal to $\max(S)$.

To see how the proof will follow, notice that the condition $\varphi(qx) = \varphi(x)$ looks very similar to a periodic function like $\sin(x)$. As $x$ tends to 0, the interval of a period becomes smaller and smaller. For example, if we let $q = 0.1$ and $x = 1, x = 0.1, x = 0.01$, we obtain the periods $(0.1, 1], (0.01, 0.1]$, and $(0.001, 0.01]$ respectively. The shape of $\varphi(x)$ is periodic, so that regardless of the length of each period, it resembles the graph in any other period; in this way, we can see that if $\varphi(x)$ is non-horizontal, then with infinitely small periods $\frac{\mathrm{d}_q}{\mathrm{d}_q x}\varphi(x)$ tends to infinity, meaning that $\varphi(x)$ is discontinuous at $x = 0$. This is the general idea of the proof.

*Proof of Theorem 3.4.* This will be a proof by contradiction. Suppose that $F_1$ and $F_2$ are two $q$-antiderivatives of $f$ such that they are both continuous at 0. Let there exist some function $\varphi = F_1 - F_2$. We let $\varphi$ be continuous at 0, and have the property $\varphi(qx) = \varphi(x)$, or periodicity, for all $x$ since $\frac{\mathrm{d}_q}{\mathrm{d}_q x}\varphi(x) = 0$. For some $A > 0$, define

$$m = \inf\{\varphi(x) | qA \leq x \leq A\},$$
$$M = \sup\{\varphi(x) | qA \leq x \leq A\}.$$

$A$ may be $\infty^+$ depending on whether $\varphi(x)$ is unbounded. In other words, $m$ is the infimum of the set of $\varphi(x)$ and $M$ is the supremum of the set of $\varphi(x)$. If we assume that $m < M$, then at least one of the statements $\varphi(0) \neq m$ and $\varphi(0) \neq M$ must be true, because they must be related to each other in some way. Suppose that $\varphi(0) \neq m$. By the definition of continuity at $x = 0$, if we have some $\epsilon > 0$ sufficiently small, then there exists $\delta > 0$ such that

$$m + \epsilon \notin \varphi(0, \delta), \tag{3.3}$$

or that $m + \epsilon$ lies in the open interval $\{\phi(x) | 0 < x < \delta\}$. We can replicate this notation for closed intervals as well. Alternatively, we have for $N$ sufficiently large $q^N A < \delta$. Then, using the periodic property of $\phi(x)$ we can write

$$m + \epsilon \in (m, M) \subset \phi[qA, A] = \phi[q^N qA, q^N A] = \phi[q^{N+1}A, q^N A] \subset \phi(0, \delta),$$

which clearly implies that $m + \epsilon \in \phi(0, \delta)$, which contradicts (3.3). We can repeat this argument when $m > M$, and we will find the same thing, so it must be true that $m = M$. Then $\phi(x)$ is constant in the not-necessarily bounded closed interval $[qA, A]$, implying that it is constant everywhere, and that our $q$-antiderivative must be unique. ∎

Now that we have shown the uniqueness of the $q$-antiderivative, it seems reasonable to determine how we compute the definite $q$-antiderivative of some function. In classical calculus, this is determined by the fundamental theorem of calculus, so we will consider its $q$-analogue. To do this, we must define an object known as the Jackson integral.

**Definition 9** (Jackson integral)**.** Let $f(x)$ be a function. Then we define the series expansion

$$F(x) = \int_0^a f(x)\mathrm{d}_q x = (1 - q)a \sum_{k=0}^{\infty} q^k f(q^k a),$$

and refer to the middle term as the Jackson integral.

Notice that this appears very similar to the Riemann sum in classical calculus. In the RHS, we are taking an infinite sum of the product $q^k f(q^k a)$, which functions as the area of a rectangle. In this way, we have defined the definite integral from 0 to $a$ of $f(x)$. We can see that its generalisation is a $q$-analogue of the Riemann sum by allowing $q = 1$. Though we have defined it, we have not shown that it is indeed a bounded series, and we will not prove that here. The proof and the generalisation of the Jackson integral can be found in [KC12]. We will conclude with the $q$-analogue of the fundamental theorem of calculus.

**Theorem 3.5** ($q$-analogue of the fundamental theorem of calculus)**.** *Let $F(x)$ be the $q$-antiderivative of $f(x)$ and continuous at $x = 0$, and also define $a, b \in [0, \infty)$. Then*

$$\int_a^b f(x)d_q x = F(a) - F(b).$$

*Proof.* Since $F(x)$ is continuous at $x = 0$, we can express $F(x)$ in terms of the Jackson formula as

$$F(x) = (1-q)a\sum_{k=0}^{\infty} q^k f(q^k a) + F(0), \tag{3.4}$$

where $F(0)$ represents the constant of integration. As defined by Definition 9, we have

$$\int_0^a f(x)\mathrm{d}_q x = (1-q)a\sum_{k=0}^{\infty} q^k f(q^k a).$$

Substituting the $q$-integral into (3.4), we can simplify to obtain

$$\int_0^a f(x)\mathrm{d}_q x = F(a) - F(0).$$

Repeating this process for a fixed $b$, we have

$$\int_0^b f(x)\mathrm{d}_q x = F(b) - F(0).$$

Subtracting, we have

$$\int_a^b f(x)\mathrm{d}_q x = \int_0^b f(x)\mathrm{d}_q x - \int_0^a f(x)\mathrm{d}_q x = F(b) - F(0) - (F(a) - F(0)) = F(b) - F(a).$$

■

We can extend this to improper integrals as well, but we will not do this here. More information on $q$-antiderivatives and the methods of integration such as the $q$-analogue of integration by parts can be found in [KC12].

## 4. $q$-Pochhammer symbols and $q$-hypergeometric functions

4.1. **The $q$-Pochhammer symbol.** To develop the notion of $q$-hypergeometric series, we will need to make use of the $q$-Pochhammer symbol. We defined the classical Pochhammer symbol earlier, so we will now define the $q$-analogue of the Pochhammer symbol.

**Definition 10** ($q$-analogue of the Pochhammer symbol)**.** We define

$$(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k).$$

If we let $a = q$, notice that we can then derive the identity

$$(q;q)_n = \prod_{k=0}^{n-1}(1-q^{k+1}) = \prod_{k=1}^{n}(1-q^k), \tag{4.1}$$

We define $(a;q)_0 = 1$. To see how the $q$-Pochhammer symbol and the $q$-factorial are related, notice that by dividing each side of 4.1 by $(1-q)^n$, we can obtain $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$. To see how the $q$-Pochhammer symbol is the $q$-analogue of the Pochhammer symbol, let $(x)_n$ be the Pochhammer symbol over $\mathbb{N}$ as described earlier. Then we can see

$$\begin{aligned}\lim_{q\to 1}\frac{(q^x;q)_n}{(1-q)^n} &= \lim_{q\to 1}\frac{(1-q^x)(1-q^{x+1})\cdots(1-q^{x+n-1})}{(1-q)^n}\\ &= (x)_n.\end{aligned}$$

The $q$-Pochhammer symbol has a number of interesting properties. We can express the $q$-binomial coefficient in terms of the $q$-Pochhammer symbol. Then we have

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Also, we can extend it to the negative integers $n$. We can express the finite product in terms of the infinite product

$$(a;q)_n = \prod_{k=0}^{\infty}\frac{1-aq^k}{1-aq^{k+n}} = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

where we extended the $q$-Pochhammer symbol to an infinite product

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

With this extension to an infinite product, we can now write, for $n \in \mathbb{Z}^+$,

$$(a;q)_{-n} = \prod_{k=0}^{-n-1}(1-aq^k) = \prod_{k=1}^{n}\frac{1}{1-\frac{a}{q^k}}.$$

We say that a ***q*-series** is an infinite series with coefficients that are the $q$-Pochhammer symbol or some expression involving the $q$-Pochhammer symbol.

4.2. **Hypergeometric functions.** We will begin by defining the classical generalised hypergeometric function.

**Definition 11** (Generalised hypergeometric function)**.** Let $(q)_n$ be the classical Pochhammer symbol, or the rising factorial as defined earlier. For $|z| < 1$, we define the hypergeometric function by the power series

$$\begin{aligned}{}_rF_s[a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;z] &= \sum_{n=0}^{\infty}\left(\frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}(b_j)_n}\right)\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\prod_{k=0}^{n-1}\frac{\prod_{i=1}^{r}(a_i+k)}{\prod_{j=1}^{s}(b_j+k)}\right)\frac{z^n}{n!}.\end{aligned}$$

This is the generalised hypergeometric function. Though this expression looks very complex, we can decompose to understand its different components. From its definition, we can see that it is an infinite series, for which the two lists $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ as well as the variable $z$ are its inputs. The expression within the product is in fact a rational function. This leads us to a property of hypergeometric functions. The coefficient $R(n)$ is given by the ratio $c_{n+1}/c_n$ which is a rational function, and we say that $c_0 = 1$. An input list that is the empty set is notated as "$-$". If we consider different values of $r$ and $s$, we can produce several well-known functions. For instance,

$$ {}_0F_0[-;-;z] = \sum_{n=0}^{\infty} \frac{z^n}{n!}, $$

which is the Taylor series expansion for $e^z$. Similarly,

$$ {}_1F_0[a;-;z] = \sum_{n=0}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)}{n!} z^n = \frac{1}{(1-z)^a}, $$

which is a geometric series. As can be seen, the generalised hypergeometric function allows us to derive a number of different special series, and acts as a general function for all series of this type. The most common hypergeometric series is the function ${}_2F_1[a,b;c;z]$, and is sometimes simply denoted as $F(z)$ for the sake of simplicity. When looking at the $q$-analogue of the hypergeometric series, we will be looking at a special case to derive an important result.

4.3. **$q$-hypergeometric functions.** The $q$-analogue of the hypergeometric series is sometimes known as the basic hypergeometric series.

**Definition 12** ($q$-analogue of the generalised hypergeometric function)**.** Let $(a;q)_n$ denote the $q$-Pochhammer symbol. We define the $q$-analogue of the generalised hypergeometric function to be

$$ {}_r\Phi_s[a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q; z] = \sum_{n=0}^{\infty} \left( \frac{\prod_{i=1}^{r}(a_i;q)_n}{\prod_{j=1}^{}(b_j;q)_n} \right) \left( (-1)^n q^{\frac{n(n-1)}{2}} \right)^{s-r+1} z^n. $$

We can see that this definition satisfies the conditions of being a $q$-analogue because when we let $q$ approach 1, the resulting expression is identical to the definition of the generalised hypergeometric function. Similar to the generalised hypergeometric function, the basic hypergeometric function can become different series given the correct values for $r$ and $s$. In this way, it is a generalisation of the $q$-analogues of many different well-known functions. Let us consider the case where $r = 1$ and $s = 0$. Then we have

$$ {}_1\Phi_0[a;-;q;z] = \sum_{n=0}^{\infty} \frac{(1-a)_q^n}{(1-q)_q^n} z^n. $$

Let $a = q^N$, where $N \in \mathbb{Z}^+$. Then

$$ {}_1\Phi_0[q^N;-;q;z] = \sum_{n=0}^{\infty} \frac{(1-q^N)_q^n}{(1-q)_q^n} z^n = \sum_{n=0}^{\infty} \frac{(1-q^N)(1-q^{N+1})\cdots(1-q^{N+n-1})}{(1-q)(1-q^2)\cdots(1-q^{n-1})} z^n. $$

Clearly, from the definition of the $q$-factorial, this can be simplified as

$$ \sum_{n=0}^{\infty} \frac{[N]_q[N+1]_q\cdots[N+n-1]_q}{[n]_q!} z^n. $$

This is the $q$-Taylor series expansion of $1/(x-a)_q^n$, so by Proposition 2, this implies that

$$ {}_1\Phi_0[a;-;q;z] = \frac{1}{(z-a)_q^n}. \tag{4.2} $$

With this result, we can now confront a very interesting result due to Heine.

**Theorem 4.1.** *For any $a$,*

$$ {}_1\Phi_0[a;-;q;z] = \frac{(1-ax)_q^\infty}{(1-x)_q^\infty}. $$

*Proof.* To begin, notice that if we let $a = q^N$, then we have the expression

$$ \begin{aligned} \frac{(1-q^N z)_q^\infty}{(1-z)_q^\infty} &= \frac{(1-q^N z)(1-q^{N+1}z)\cdots}{(1-z)(1-qz)\cdots(1-q^{N-1}z)(1-q^N z)(1-q^{N+1}z)\cdots} \\ &= \frac{1}{(1-z)_q^N}. \end{aligned} $$

By (4.2), this is equivalent to ${}_1\Phi_0[a;-;q;z]$. We can now express the LHS and RHS in terms of series with coefficients $c_n(a)$ and $c'_n(a)$

$$ {}_1\Phi_0[a;-;q;z] = \sum_{n=0}^{\infty} c_n(a) z^n \quad \text{and} \quad \frac{(1-ax)_q^\infty}{(1-x)_q^\infty} = \sum_{n=0}^{\infty} c'_n(a) z^n. $$

These coefficients are each rational polynomials in terms of $a$. From the statement we proved earlier, we know that there are an infinite number of values of $a$ at which these two series share the same coefficient, or where $c_n = c'_n$. In fact, this point is exactly $a = q^N$, where $N \in \mathbb{Z}^+$. Since there are an infinite number of intersection points, the rational polynomial $c_n - c'_n$ must have an infinite number of zeroes. However, such a rational polynomial does not exist, as the degree of the denominator cannot be greater than the degree of the numerator. Thus $c_n - c'_n$ must be identically zero, meaning that we must have $c_n = c'_n$. This shows that the two series must be equivalent, so that the two expressions are equivalent. ■

Another fact of the basic hypergeometric function is that it can provide an alternative form for the $q$-analogue of the binomial theorem that we proved several times earlier.

**Theorem 4.2** ($q$-analogue of the binomial theorem in terms of the basic hypergeometric function)**.** *For any $a$,*

$$ {}_1\Phi_0[a;-;q;z] = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. $$

The second term is in fact a $q$-series, with a coefficient that is a basic hypergeometric function. The proof of this incredible result may be found in [GR04].

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