Elementary Algebra of Hyperreals

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Three Axioms and $\mathbb{R}^*$

Axiom A

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- *Complete*: Basically, there exists a metric over $\mathbb{R}$. e.g. over $\mathbb{R}$ we have the Euclidean metric $|x - y|$.
- *Ordered*: There exists an ordering of elements of the field. For $a, b, c \in \mathbb{R}$ (where our field in question is $(\mathbb{R}, +, \times)$),
  - $a \leq b \implies a + c \leq b + c$,
  - $(0 \leq a \wedge 0 \leq b) \implies 0 \leq a \cdot b$.

Axiom B

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$\mathbb{R}^*$, which denotes the set of hyperreal numbers, is an ordered field extension of $\mathbb{R}$.

- *Field extension*: Informally, $\mathbb{R}^*$ "extends" $\mathbb{R}$ to a "larger" set. Formally, $\mathbb{R}$ is a *subfield* of $\mathbb{R}^*$, where $\mathbb{R}^*$ is the *extension field* and retains properties of the field $\mathbb{R}$.

Axiom C

Axiom

$\mathbb{R}^*$ has a positive *infinitesimal*, i.e. there exists an element $\varepsilon \in \mathbb{R}^*$ such that $\varepsilon > 0$ and $\varepsilon < r$ for every $r \in \mathbb{R}^+$.

- Recall this from our initial treatment of infinitesimals.

Defining $\mathbb{R}^*$

Definition

Elements of $\mathbb{R}^*$

- Some $x \in \mathbb{R}^*$ is
  - *positive infinitesimal* if $|x| < r$ for all $r \in \mathbb{R}^+$,
  - *finite* if $|x| < r$ for any $r \in \mathbb{R}$,
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  - *finite* if $|x| < r$ for any $r \in \mathbb{R}$,
  - *infinite* if $|x| > r$ for all $r \in \mathbb{R}$.
- Furthermore, $x, y \in \mathbb{R}^*$ are *infinitely close* (written $x \approx y$) if $x - y$ is infinitesimal. This implies that x is infinitesimal iff $x \approx 0$.
- The only real infinitesimal is 0.

Monads and galaxies

The following allow us to describe certain sets of hyperreals.

Definition (Monad)

For some $x \in \mathbb{R}^*$, we define the *monad of* x to be

$$\text{monad}(x) = \{y \in \mathbb{R}^* | x \approx y\}.$$

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Definition (Galaxy)

For some $x \in \mathbb{R}^*$, we define the *galaxy of* x to be

$$\mathrm{galaxy}(x) = \{y \in \mathbb{R}^* | x - y \text{ is finite}\}.$$

Thus monad(0) is the set of infinitesimals and galaxy(0) is the set of finite hyperreal numbers.

The Algebraic Structure of $\mathbb{R}^*$

Describing galaxies

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Theorem

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Corollary

Any two galaxies are either equal or disjoint.

- Means that all galaxies are unique or identical. Behaves like residues, i.e. galaxy(x) is the coset of x modulo galaxy(0), or

$$\text{galaxy}(x) = \{x + a | a \in \text{galaxy}(0)\}.$$

Bringing monad(0) into the picture

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Any two monads are either equal or disjoint.

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Corollary

Any two monads are either equal or disjoint.

- "infinite closeness" or $x \approx y$ forms an equivalence relation over $\mathbb{R}^*$.
- The proof follows similarly to galaxy(0). The equivalence relation can be shown by proving reflexivity, symmetry, and transitivity.

A powerful result

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- A direct corollary is that there are negative infinitesimals and negative infinite elements.

Some notes

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Example

Consider
$$\varepsilon \cdot \frac{1}{\varepsilon} = 1 \notin \text{monad}(0).$$

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Corollary

Suppose x, y are finite hyperreals. Then

1. $x \approx y \iff st(x) = st(y)$,
2. $x \approx st(x)$,
3. *if* $r \in \mathbb{R}$ *then* $st(r) = r$,
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- Do the proofs yourselves.

Another theorem

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The standard part function is a ring homomorphism of the ring galaxy(0) *onto the field of real numbers. For finite* x, y,

1. $st(x \oplus y) = st(x) \oplus st(y)$,
2. $st(x \otimes y) = st(x) \otimes st(y)$.

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- Sounds more complicated than it is. A ring homomorphism is a structure-preserving map that takes elements of galaxy(0) to elements of $\mathbb{R}$ using the properties of the ring, which in this case are $\otimes$ and $\oplus$.

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- Sounds more complicated than it is. A ring homomorphism is a structure-preserving map that takes elements of galaxy(0) to elements of $\mathbb{R}$ using the properties of the ring, which in this case are $\otimes$ and $\oplus$.
- The proof is also pretty straightforward.

More interesting stuff that Keisler covers

- Model theory is incredibly important to studying hyperreal numbers.
- Keisler covers a lot more rigorously in [1].

H. Jerome Keisler.
Foundations of Infinitesimal Calculus.
University of Wisconsin, 2000.
Available online at
https://www.math.wisc.edu/~keisler/foundations.html.

Thanks

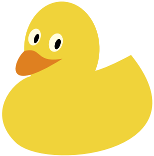

Figure: The duck thanks you.