Omnific Integers

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The Omnific Integers

Definition

We define an omnific integer $x \in \mathbb{OZ}$ to be such that

$$x = (x - 1 | x + 1).$$

1. $\mathbb{Z}_{\text{No}}$ (1,-1,2,31415926535897932384626),
2. $\omega, \omega + 1, \sqrt{\omega}, \omega^2$, etc.

It can be shown that all ordinals are omnific integers. As a corollary to this fact, every number x can be represented as a quotient of two omnific integers.

The Omnific Integers (continued)

We can show that the omnific integers are closed under addition (naturally, subtraction) and multiplication. It is sufficient to show that if $x, y \in \mathbb{OZ}$, then $x \pm y \in \mathbb{OZ}$ and $xy \in \mathbb{OZ}$.

Exponentiation in $\mathbb{Z}_{\text{No}}$

Theorem

Let some $\alpha \in \mathbb{Z}^+_{No}$, where $\alpha = (x|\phi)$ and $x \in \mathbb{Z}^+$ (i.e. a positive integer surreal). Then

$$\alpha^n = \left(n\alpha^{n-1}x - \sum_{r=2}^{n-1} \binom{n}{r} \alpha^{n-r} x^r - x^n \middle| \phi \right)$$

when $n > 2$.

Exponentiation in $\mathbb{Z}_{\mathrm{No}}$ (continued)

Theorem

Analogously, let $\alpha \in \mathbb{Z}^-_{No}$*, where* $\alpha = (\phi| - x)$ *where* $x \in \mathbb{Z}^+$*. Then*

$$\alpha^n = \begin{cases} \left(n\alpha^{n-1}x - \sum_{r=2}^{n-1} \binom{n}{r} \alpha^{n-r} x^r - x^n \middle| \phi \right) & n \equiv 0 \pmod 2, \\ \left(\phi \middle| x^n + \sum_{r=2}^{n-1} \binom{n}{r} \alpha^{n-r} x^r - n\alpha^{n-1} x \right) & n \equiv 1 \pmod 2, \end{cases}$$

where $n > 2$*.*

Prime Factorisation in $\mathbb{Z}_{\mathrm{No}}$

We can write in general the prime factorisation of an integer surreal. If some p is prime in $\mathbb{Z}$, then its prime representation in $\mathbb{Z}^{+}_{\mathrm{No}}$ is $(p-1|\phi)$. Let $p_1, p_2, \ldots, p_2$ represent positive integer surreals. Let $\pi_1, \pi_2, \ldots, \pi_n$ represent the powers of each prime. Then for some integer surreal $x \in \mathbb{Z}_{\mathrm{No}}$, we can write

$$
\begin{aligned}
x &= p_1^{\pi_1} p_2^{\pi_2} \cdots p_{n-1}^{\pi_{n-1}} p_n^{\pi_n} \\
&= \left(\pi_1 p_1^{\pi_1 - 1} x - \sum_{r=2}^{\pi_1 - 1} \binom{\pi_1}{r} p_1^{\pi_1 - r} x^r - x^{\pi_1} \middle| \phi \right) \cdot \\
&\left(\pi_2 p_2^{\pi_2 - 1} x - \sum_{r=2}^{\pi_2 - 1} \binom{\pi_2}{r} p_2^{n - r} x^r - x^{\pi_2} \middle| \phi \right) \cdots \\
&\left(\pi_n p_n^{\pi_n - 1} x - \sum_{r=2}^{\pi_n - 1} \binom{\pi_n}{r} p_n^{\pi_n - r} x^r - x^{\pi_n} \middle| \phi \right).
\end{aligned}
$$

Classification of Elements of $\mathbb{OZ}$

1. Conway also establishes a division algorithm.

Theorem

If a and b are integers with $b > 0$, then there exist unique omnific integers q and r such that $a = bq + r$, where $0 \leq r < b$.

2. A *descending chain condition* occurs in an algebraic structure when there is no infinite decreasing ordering. $\mathbb{OZ}$ does not satisfy this property. Therefore a division algorithm does not establish unique factorisation in $\mathbb{OZ}$.
3. This is illustrated by the following non-unique factorisation of ω.

Example

$$\omega = 2(\omega/2) = 3(\omega/3) = \cdots = (\sqrt{\omega})^2 = (\sqrt[3]{\omega})^3 = \cdots .$$

Classification of Elements of $\mathbb{OZ}$ (continued)

We need the following definition.

Definition (Divisible)

An omnific integer is *divisible* iff it is divisible by every finite non-zero omnific integer.

Theorem

Every omnific integer is uniquely the sum of a divisible omnific integer and a finite omnific integer.

If we restrict the integers mentioned here to ordinal numbers, then this creates a division algorithm analogous to the previous theorem with numbers of the form $a = \omega q + r$. Otherwise the two theorems are separate.

Classification of Elements of $\mathbb{OZ}$ (continued)

1. For numbers that are indivisible, meaning they are not divisible by every finite non-zero integer, there also exists a series of non-unique factorisations.
2. There is also a notion of primality.

Example

$$\omega + \sqrt{\omega} + \sqrt[3]{\omega} + \cdots + 1$$

is prime.

Survey of number-theoretic things in $\mathbb{OZ}$

1. Pell equations and continued fractions.
   A Pell equation is of the form $x^2 - Ny^2 = \pm 1$, where N is some fixed omnific integer. We can consider this in the context of omnific integers by letting x and y be omnific integers. The theory can be studied by considering *convergents* of the infinite continued fraction expansion of $\sqrt{N}$.
2. There are many analogues and trivial/nontrivial things to investigate in the omnific integers. Ex. modular arithmetic (modulo an omnific integer?), sum of two squares, literally every number-theoretic problem in existence.

Thank you!